Toward Multi-Scale Simulation of the Atmosphere and Ocean

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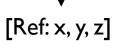


About this presentation

I have tried to annotate the presentation to point to the literature so that each part of the presentation can be studied further.

The references are listed on the last page of this PDF.

My hope is that this file will be a resource to those who develop atmosphere and ocean dynamical cores in the future.





Outline

Overview (60-90 minutes)

Basic motivation for atmosphere and ocean models

Design a dynamical core

Spanning the space

Numerical approaches to global atmosphere and ocean modeling

Evaluation metrics

A Multi-Scale Approach to Global Atmosphere and Ocean Modeling (90-120 minutes)

Motivation

Variable resolution meshes

Design of a finite-volume solver for variable resolution meshes

Results using the shallow-water equations

Results using the global hydrostatic primitive equations for atmosphere modeling

Results using the global hydrostatic primitive equations for ocean modeling

Computational considerations

Challenges in constructing a true multi-scale climate system model





Basic motivation for the design of global atmosphere and ocean models.



Modeling the global climate system is an enormous challenge.

A few of the obstacles:

- I. Wide range of spatial and temporal scales.
- 2. Interacting physical systems with widely disparate scales of motion.
- 3. Physical systems interacting with biological systems.
- 4. Biological systems interacting with anthropogenic systems.
- 5.A very incomplete knowledge of how each of these systems work.



My Goals for this morning's session

- I.A physically-based motivation for why the design of robust atmosphere and ocean dynamical core is important.
- 2. Making clear the ties to those parts of the atmosphere and ocean model that are equally important, but are not a part of the dynamical core (i.e. parameterizations).
- 3.A description of a numerical approach that allows us to capture the wide range of spatial and temporal scales in the atmosphere and ocean systems in a computationally-tractable manner.

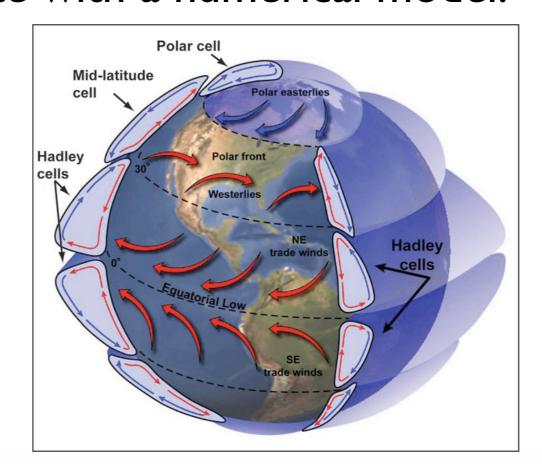


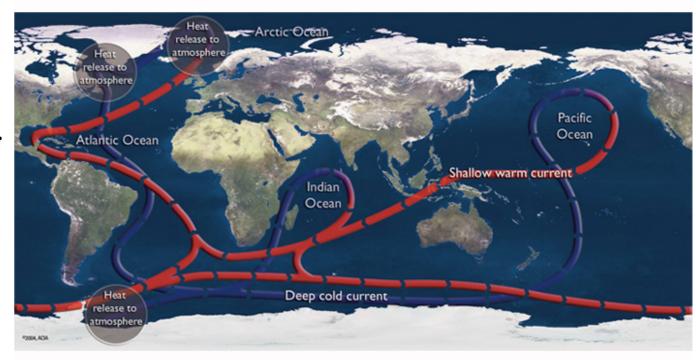


The task is simple (!?). Reproduce the observed climate with a numerical model.

At the foundation of climate dynamics is the forcing due to differential heating, so our models must have an accurate representation of the flow of energy through the climate system.

- I. correct net solar forcing in
 - a. clouds
 - b. albedo
- 2. correct net long-wave radiation out
 - a. clouds and water vapor
 - b. ocean sea-surface temperature.
- 3. correct poleward transfer of energy
 - a. synoptic weather in atmosphere
 - b. eddies and boundary currents in ocean.
- 4. correct transfer of energy between systems.
 - a. atmosphere/ocean coupling
 - b. atmosphere/sea ice
 - b. ocean/sea ice coupling.









Dynamical cores as simulators of the general circulation.

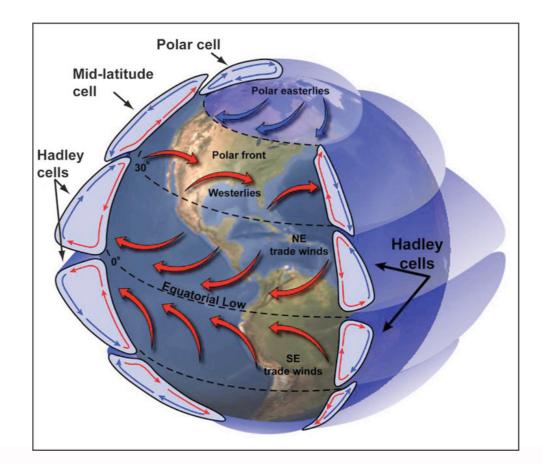
At the heart of atmosphere and ocean models is what we refer to as the dynamical core.

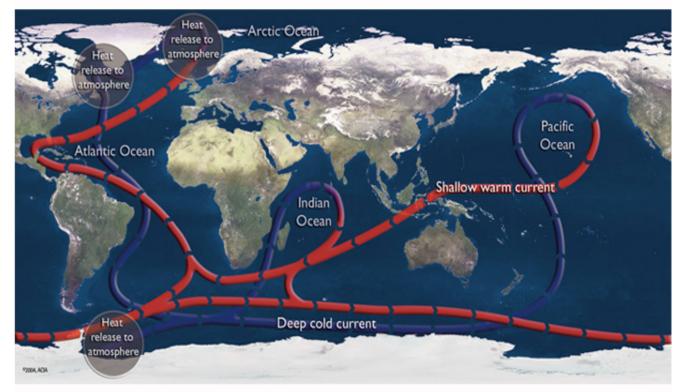
The dynamical core is responsible for evolving the distribution of mass, momentum, energy and tracer species (e.g. water) as a function of space and time.

In the broadest sense, dynamical cores are responsible for simulating the large-scale general circulation of the atmosphere and ocean.

With respect the atmosphere, this means an accurate representation of the Hadley, Ferrel and polar cells along with an accurate representation of tropical dynamics and storm tracks.

With respect to the ocean, this means an accurate simulation of poleward heat transport in the western boundary currents, an accurate representation of water mass and their maintenance, correct deep water formation along with robust tropical dynamics.

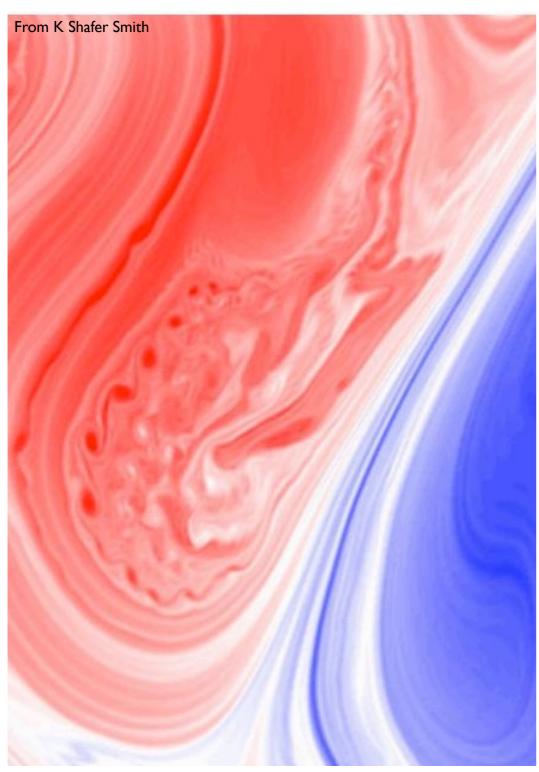




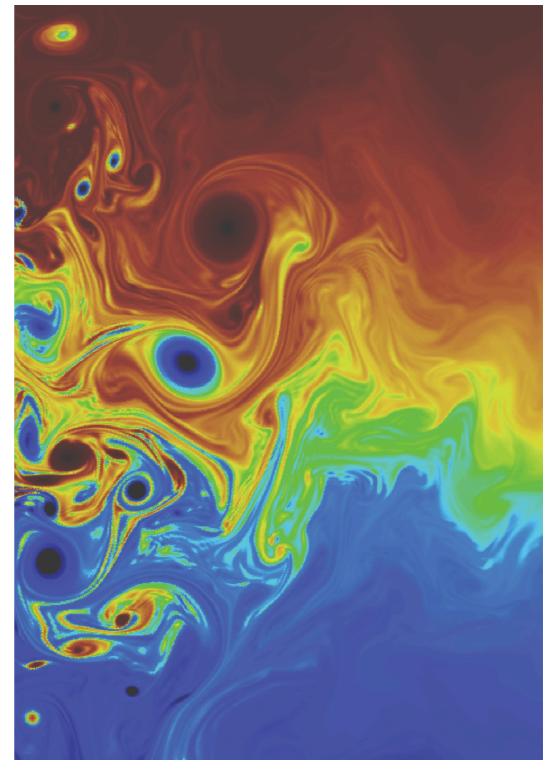




Dynamical cores as turbulence models.



Simulation of baroclinic eddies in the atmosphere.



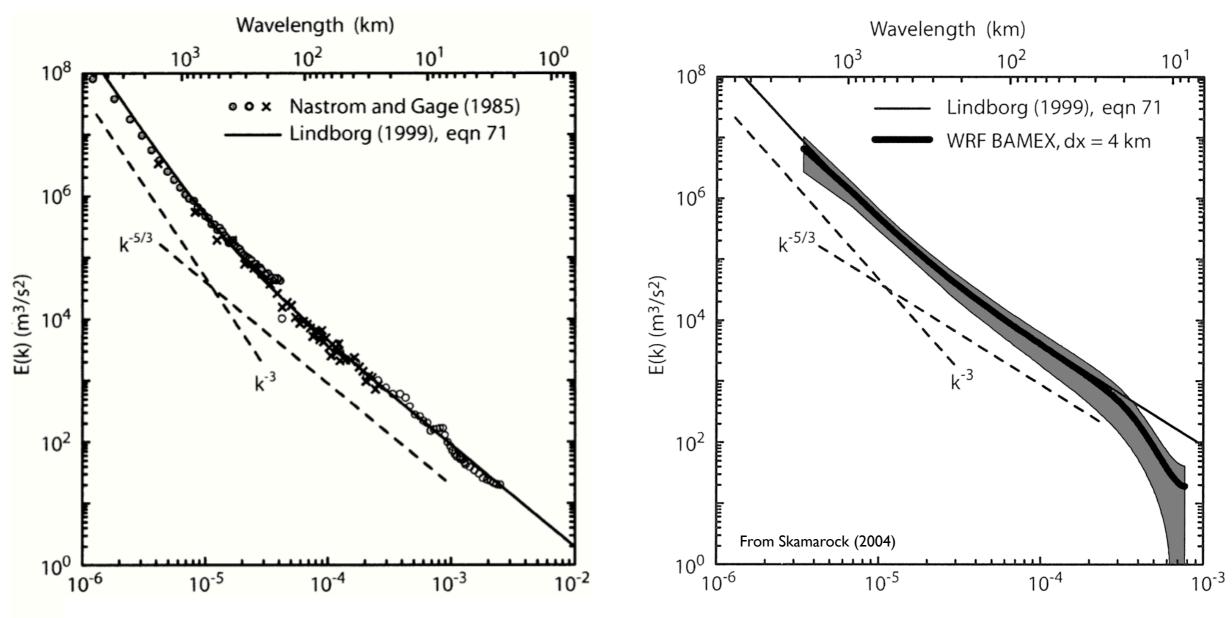
Simulation of ocean eddies.

Shown above: total potential vorticity.



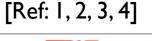


Dynamical cores as turbulence models.



Both the atmosphere and ocean fluids are "typical" nonlinear systems a broad, monotone energy spectrum. The spectrum develops and is maintained by non-local wave-wave interactions.

Dynamical cores are responsible for reproducing this spectrum from the global scale to the grid scale. An accurate representation of the global scale is best achieved through close attention to the simulation at the grid scale (!).





Dynamical cores as an applied math exercise in solving partial differential equations (PDEs).

Relative to other areas of climate modeling, dynamical cores are fortunate to have a strong theoretical foundation that yields a governing set of PDEs.

Assuming (!) perfect knowledge of the source terms (R, F and Q), the problem can be approached from a pure applied math perspective.

While this line of thinking might be a bit naive, it is strictly correct so long as we respect the invariants of the PDE system, such as conservation of mass, potential vorticity and energy.

In the climate modeling, the respect to system invariants is sometimes called the "Arakawa-approach", in other fields it often referred as the "mimetic approach."

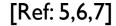
Typical formulation of the primitive equations.

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = R$$

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F}$$

$$c_v \frac{dT}{dt} + p \frac{d\alpha}{dt} = Q$$

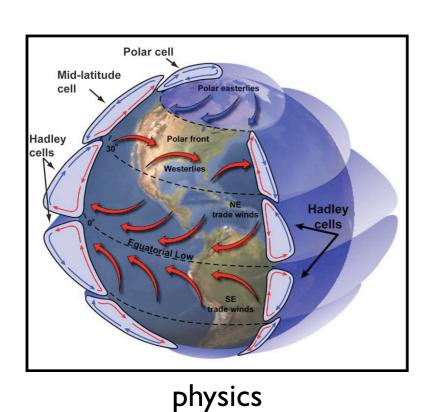
$$\rho = f(p, T,)$$

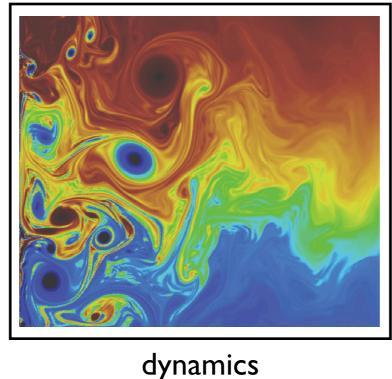




Hallmarks of a Robust Dynamical Core

- I. can reproduce the general circulation
- 2. can represent the turbulent nature the geophysical flows
- 3. respects the properties of the underlying PDEs





$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = R$$

$$\frac{d\mathbf{u}}{dt} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{F}$$

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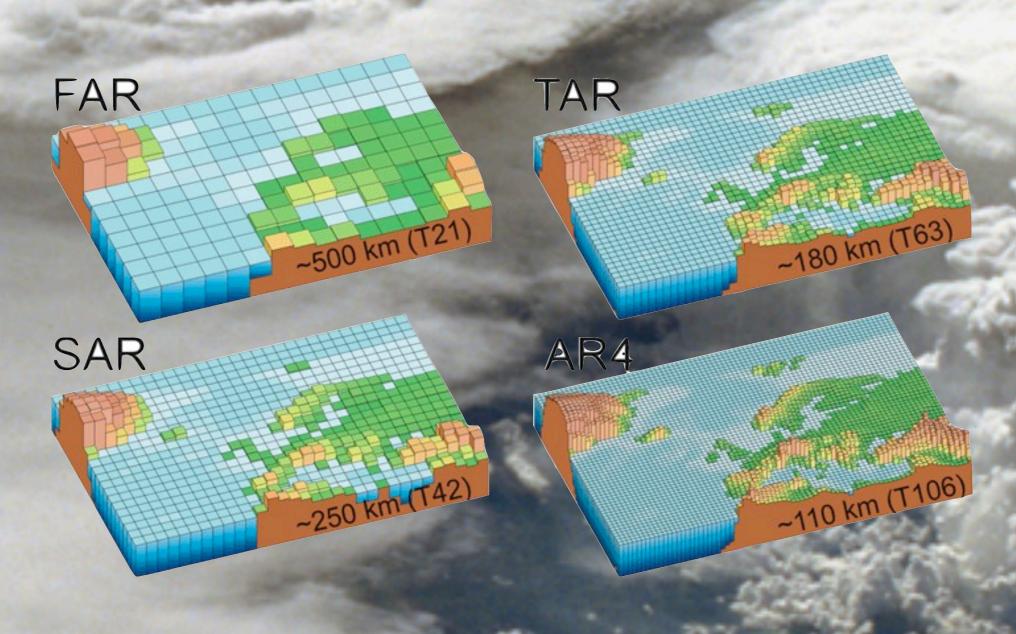
mathematics

So what is the problem?



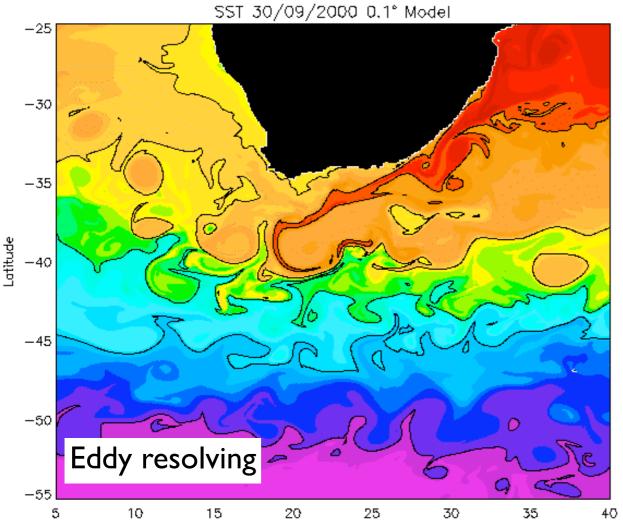


From the First Assessment Report (FAR) in 1990 through Assessment Report 4 (AR4) in 2007, resolution has improved by a factor of **5**. Another factor of **100** increase in resolution and we will be able to resolve (some) clouds!



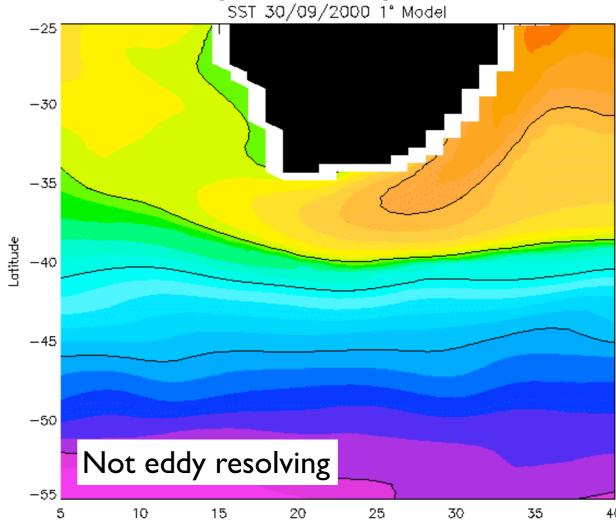
The situation is only slightly better for ocean eddies.

Aghulhas region as simulated with a global 0.1 degree model.



This is the resolution used decadal-long simulations.

Aghulhas region as simulated with a global 1.0 degree model.



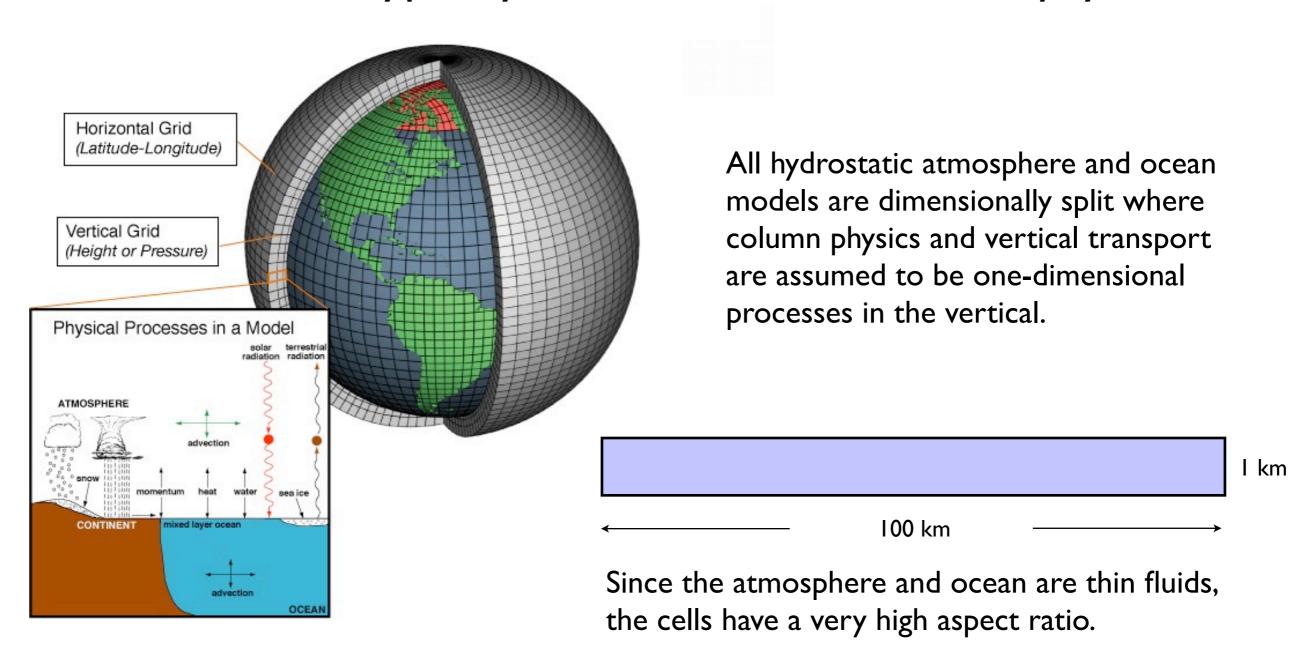
This is the resolution used in century-long climate change simulations.

When eddies are not resolved, as in typical IPCC simulations to assess anthropogenic climate change, the ocean model is obligated to attempt to determine the net impact of the eddies (if they were simulated!) on the ocean circulation based on the resolved large-scale features.





As a result of the typically large spatial and temporal gap between the large-scales resolved by the dynamical core and the small-scales parameterized by the physics, climate models are typically constructed with column physics.







OK, how do we build a dynamical core?



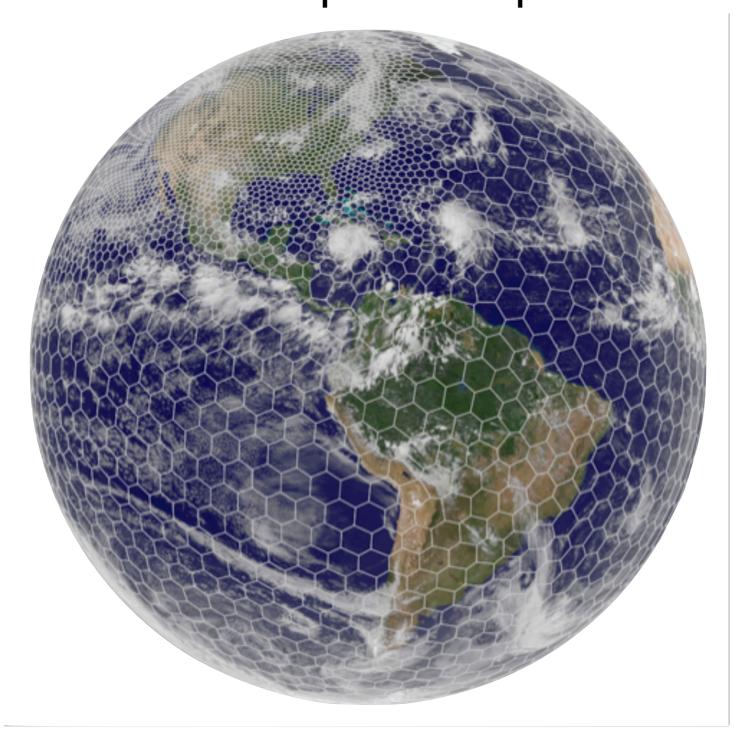
Step #1:

Span the Space.

(Not as easy as it would seem.)



If we are going to construct a numerical model, at some point the infinite degrees of freedom in the continuous system will be truncated to a finite number of degrees of freedom. We need to make sure that this finite set spans the space of interest.

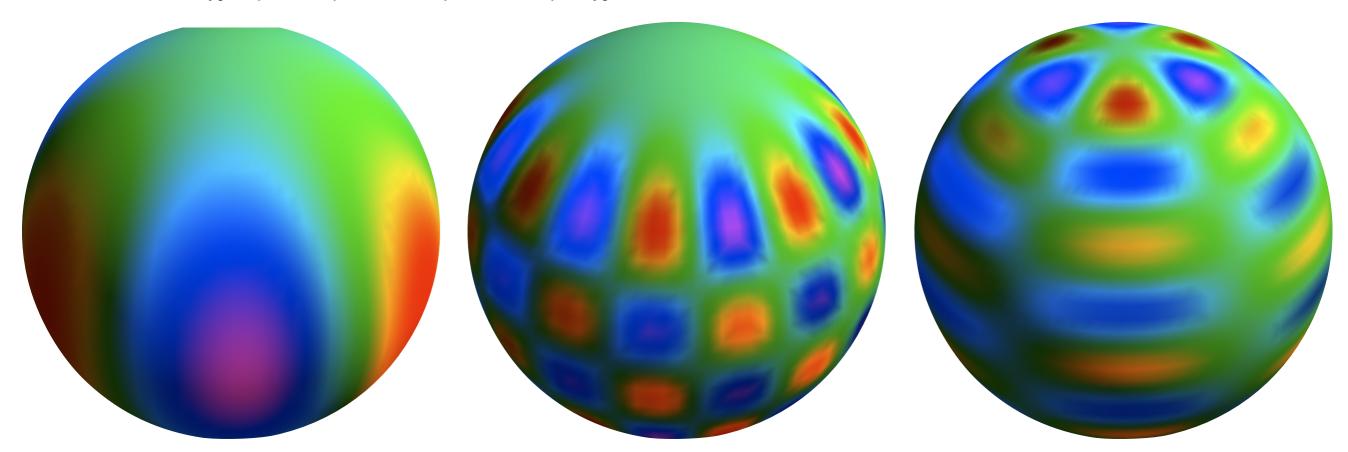




The leading approach over the last several decades has been a global spectral approach.

The set of spherical harmonics are orthogonal. We assume that our solution lives in the space of harmonics that we retain.

$$Y_n^m(\theta,\varphi) = P_n^m(\cos\theta)e^{im\varphi}, \quad n \ge 0, \ |m| \le n,$$
$$-\Delta Y_n^m(\theta,\varphi) = n(n+1)Y_n^m$$



Above shows the patterns of a few of the spherical harmonics.

[Ref: 8]



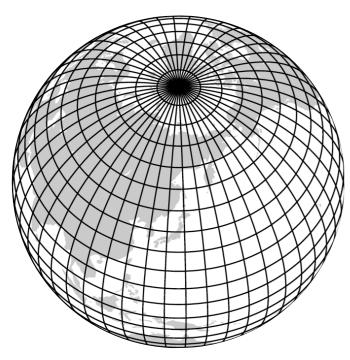


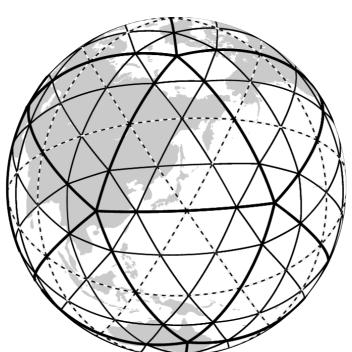
A few "grid-based" ways to span the sphere

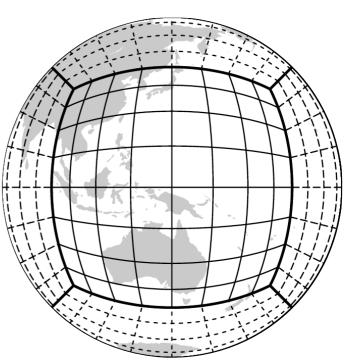
Latitude-Longitude Grid

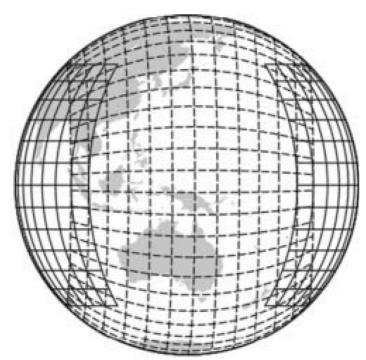
Delaunay Triangulation from inscribed icosahedron

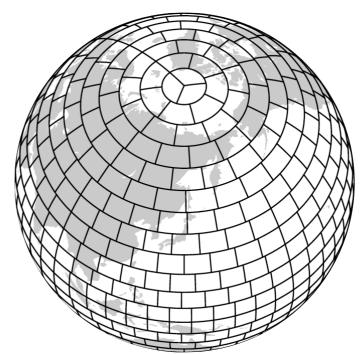
Quadrilateral Mesh from inscribed cube











Overlapping Yin-Yang Grid

Reduced Latitude-Longitude Grid

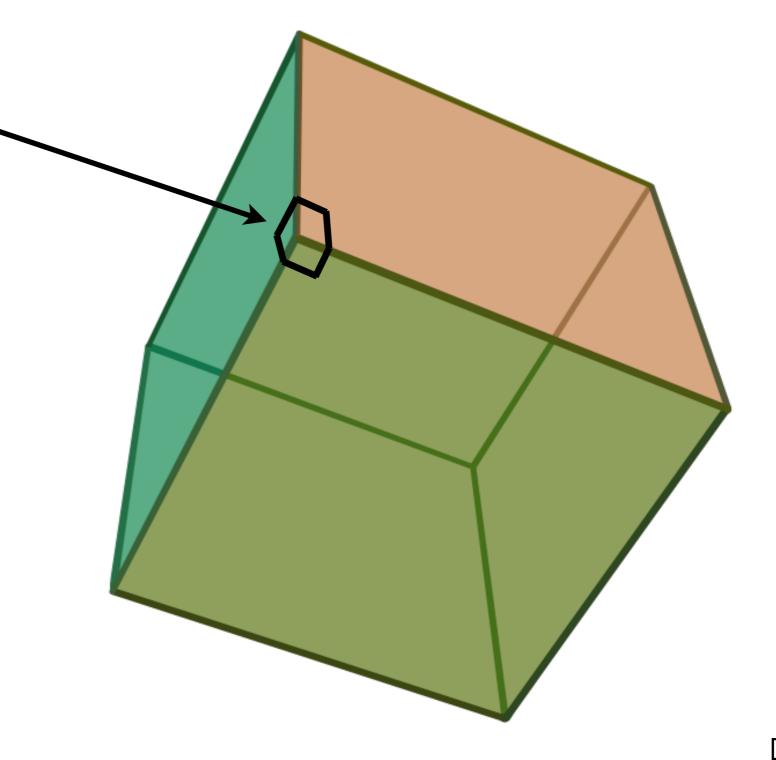




It is safe to say there this is no perfect way to mesh the sphere: angular deficiency and singularities.

Each corner is composed of three right angles (summing to 270 degrees). When projected to sphere, this corner will span 360 degrees. Each corner has an angular deficiency of 90 degrees.

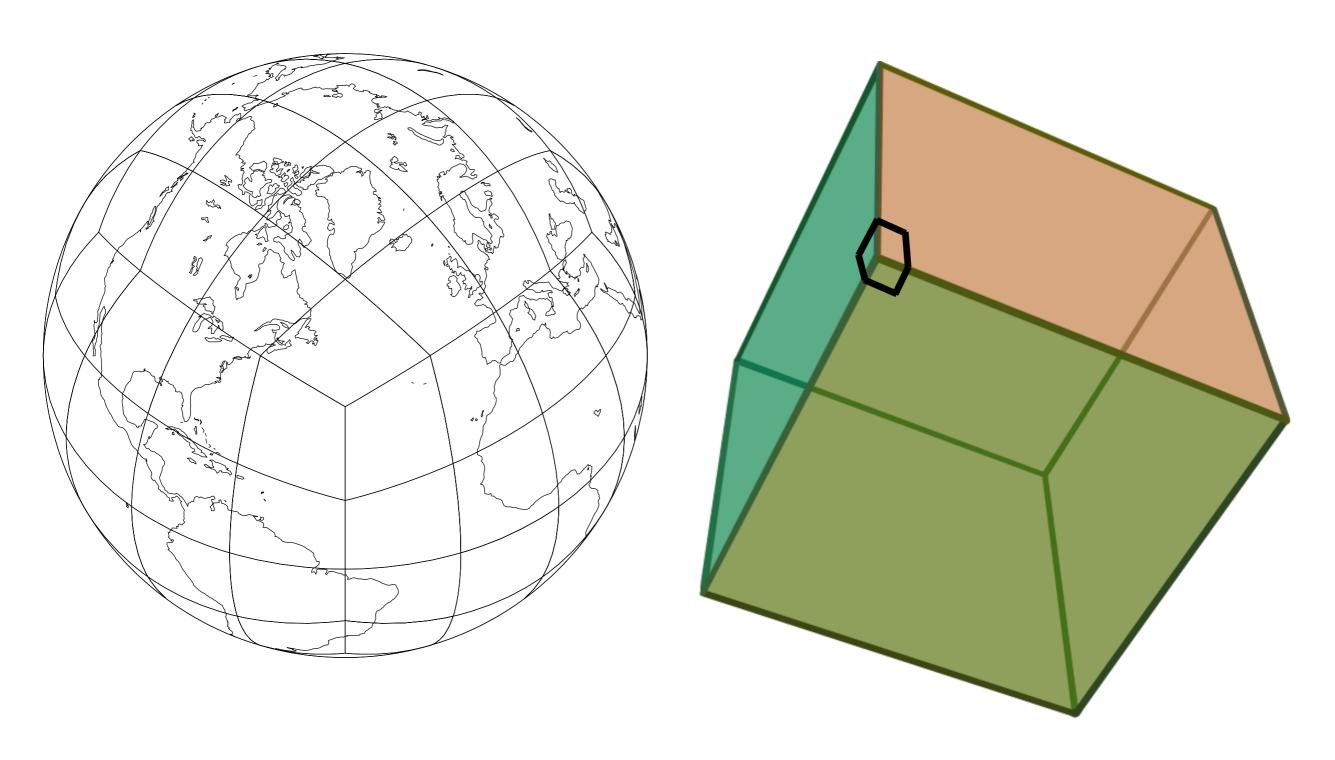
Regardless of the Platonic solid we choose, the total angular deficiency is 720 degrees. More corners imply a less severe singularity at each corner.







These singularities are regions of locally-high truncation error because the mesh is less uniform. The indexing is also different at this locations.







OK, so say that you have picked a meshing strategy for the sphere. Now you have to pair that mesh with a method to solve the PDEs.**

Step #2:

Numerical methods to global atmosphere and ocean climate modeling.

** The Mesh and the Method should not be independent choices! see http://public.lanl.gov/ringler/files/momentum.pdf for a full discussion.

[Ref: 11]





Methods for Consideration

Finite-Difference

0

Typically solves strong form of governing equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0$$

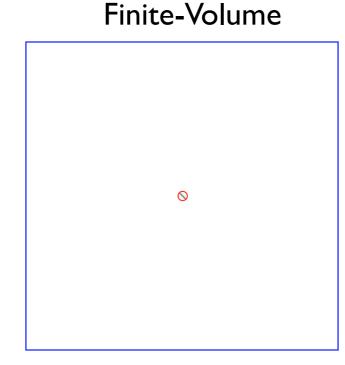
Grid-point centric. Equations evolved approximating operators at nodal locations.

Uses large stencil to approximate operators.

Methods for Consideration

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Grid-point centric.
Equations evolved approximating operators at nodal locations.

Uses large stencil to approximate operators.

Always solves weak form of governing equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \left(\rho f\right)}{\partial t} + \nabla \cdot \left(\rho f \mathbf{u}\right) = 0$$

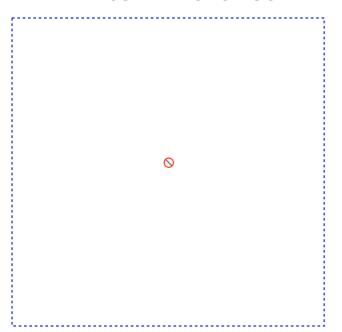
Cell-centric approach.
Equations evolved by computing fluxes across cell boundaries.

Uses compact or large stencil to approximate values along cell boundaries.



Methods for Consideration

Finite-Difference



Typically solves strong form of governing equations.

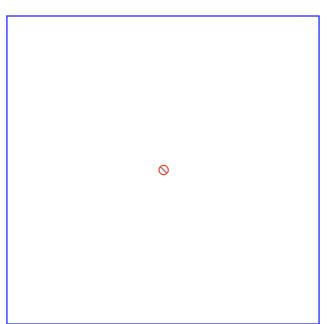
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

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Grid-point centric.
Equations evolved approximating operators at nodal locations.

Uses large stencil to approximate operators.

Finite-Volume



Always solves weak form of governing equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

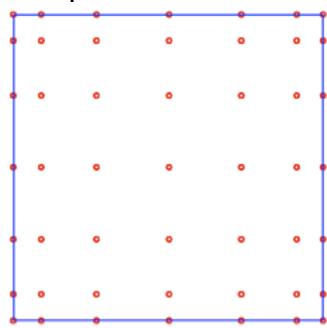
$$\frac{\partial \left(\rho f\right)}{\partial t} + \nabla \cdot \left(\rho f \mathbf{u}\right) = 0$$

Cell-centric approach.

Equations evolved by computing fluxes across cell boundaries.

Uses compact or large stencil to approximate values along cell boundaries.

Spectral Element



Can solve either the weak or strong form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \left(\rho f\right)}{\partial t} + \nabla \cdot \left(\rho f \mathbf{u}\right) = 0$$

An element-centric approach. Solution space is assumed on each element by a choice in the basis.

Along element boundaries solution can be enforced to be continuous or allowed to be discontinuous.



Evaluation Metrics for Dynamical Cores: How do we know when we have done well?





- $ilde{L}$ discrete PDE
- \hat{u} discrete PDE solution
- \hat{f} discrete PDE forcing



[Ref 15, 16]

$$\tilde{L}\left(\hat{u}
ight)=\hat{f}$$
 — This is the PDE system that we want to solve.

- \widetilde{L} discrete PDE
- \hat{u} discrete PDE solution
- \hat{f} discrete PDE forcing

Truncation Error:
$$\ \, \tilde{L}\left(\hat{u}
ight) - \hat{f} = \tilde{ au} \,$$

Apply discrete operator to known data and subtract analytic result that has been projected to grid and compute (truncation) error. Repeat for a series of meshes to measure the order of accuracy of discrete operators.





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Solution Error:
$$\tilde{L}^{-1}\left(\hat{f}\right)-\hat{u}=\tilde{\sigma}$$

Solve discrete PDE system based on analytic forcing projected onto grid and subtract the known reference solution to compute the (solution) error. Repeat for a series of meshes to measure the order of accuracy of the discrete PDE solver.





 $ilde{L}$ discrete PDE

 \hat{u} discrete PDE solution

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Apply discrete operator to known data and subtract analytic result that has been projected to grid and compute (truncation) error. Repeat for a series of meshes to measure the order of accuracy of discrete operators.

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Solve discrete PDE system based on analytic forcing projected onto grid and subtract the known reference solution to compute the (solution) error. Repeat for a series of meshes to measure the order of accuracy of the discrete PDE solver.

The operator order-of-accuracy is a lower bound on the solution order of accuracy. Keep in mind that what we really care about is the solution error.

[Ref 15, 16]





Shallow-water equations written in vorticity-divergence form.

$$\frac{\partial \delta}{\partial t} + f\zeta + g\nabla^2 h = 0$$

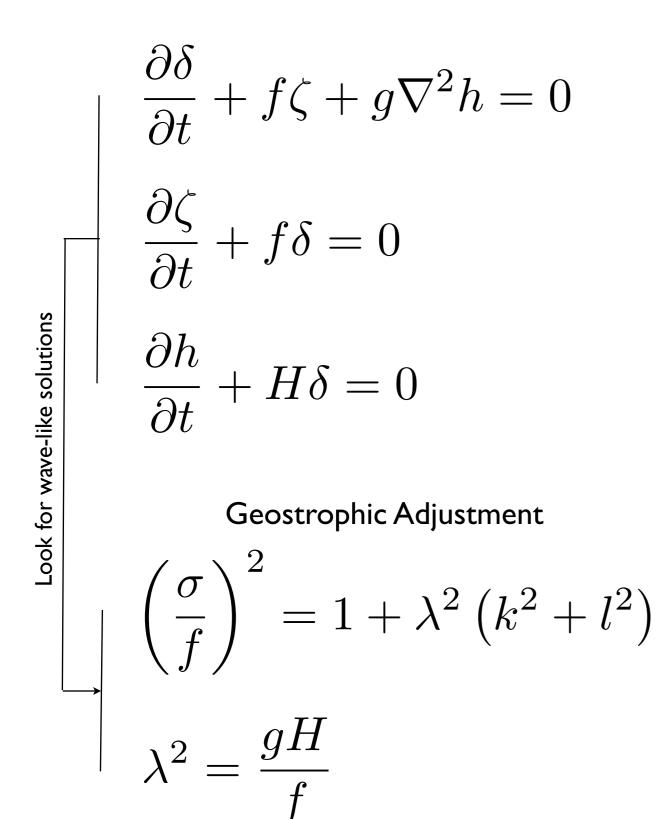
$$\frac{\partial \zeta}{\partial t} + f\delta = 0$$

$$\frac{\partial h}{\partial t} + H\delta = 0$$



[Ref: 17, 18, 19]

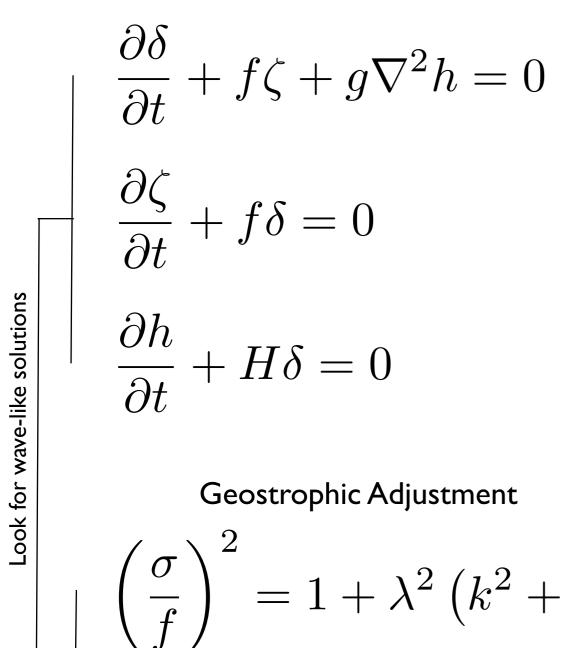
Shallow-water equations written in vorticity-divergence form.





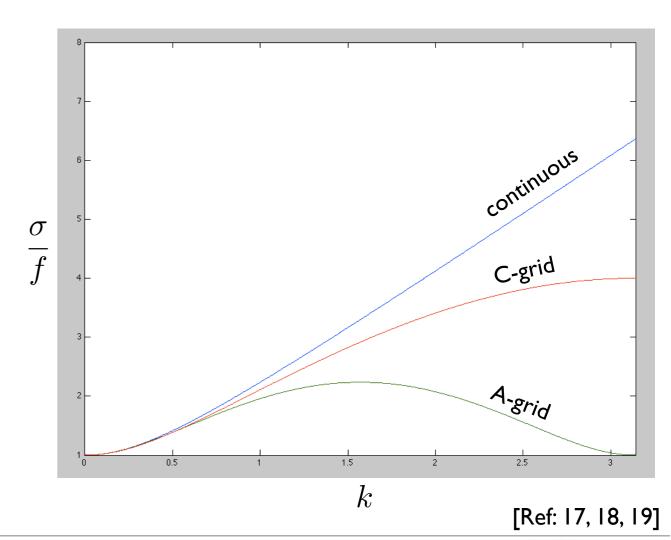
[Ref: 17, 18, 19]

Shallow-water equations written in vorticity-divergence form.

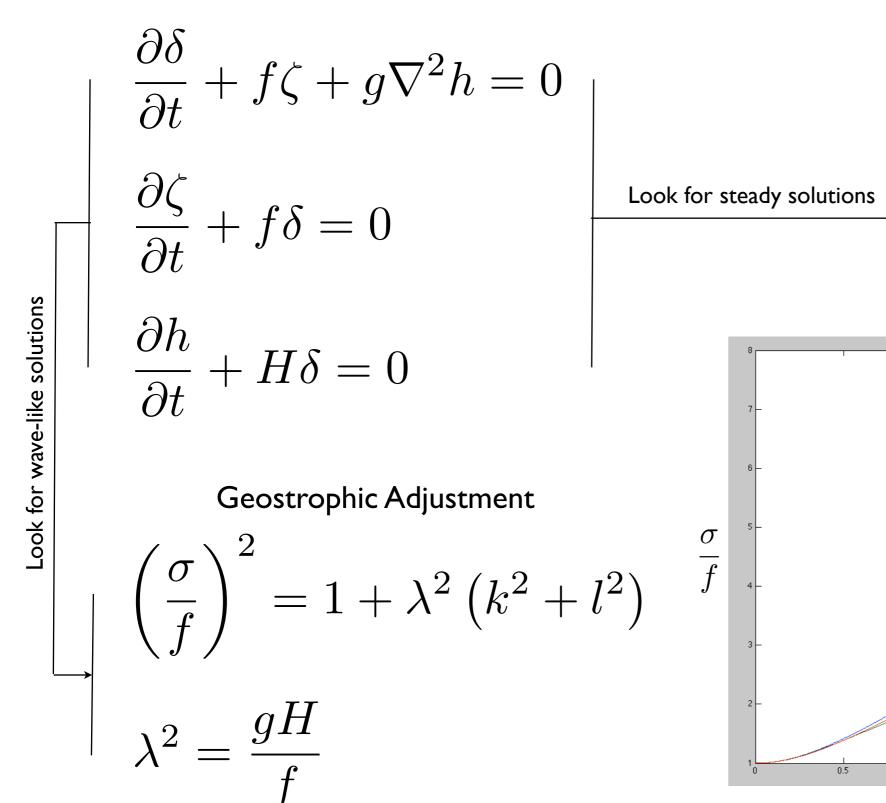


$$\left(\frac{\sigma}{f}\right)^2 = 1 + \lambda^2 \left(k^2 + l^2\right)^{\frac{\sigma}{f}}$$

$$\lambda^2 = \frac{gH}{f}$$



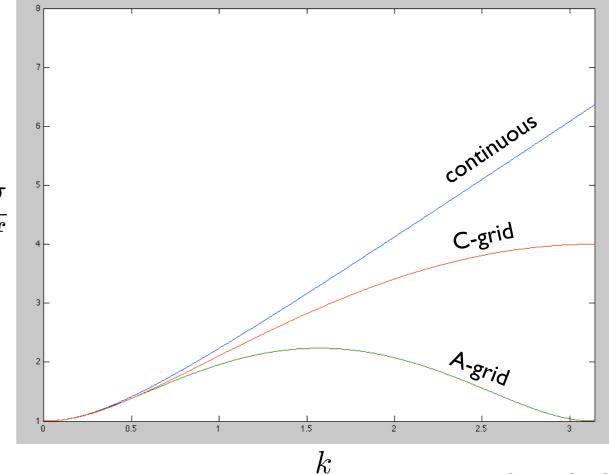
Shallow-water equations written in vorticity-divergence form.



Geostrophic Modes

$$f\zeta + g\nabla^2 h = 0$$

$$\frac{\partial(\cdot)}{\partial t} = 0$$



[Ref: 17, 18, 19]



Evaluation Metric #2: Local and Global Conservation

In frictionless, adiabatic flow the governing PDEs (either idealized 2D shallow-water or 3D primitive equations) contain a set of global invariants, i.e. quantities that remain constant over the course of a simulation.

For the 2D shallow-water system these invariants include volume, potential vorticity, total energy and potential enstrophy (to name a few).

Over the last decade the discussion of conservation has become more nuanced (rather than dogmatic), for example see [20].

With regards to potential vorticity, does the curl of the momentum lead to spurious generation of vorticity?

With regards to energy, is the flow of energy between it potential and kinetic forms conservative?





Evaluation Metric #3: Local and Global Conservation: Can we make our discrete numerical model do this?

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0, \quad q = \frac{(f + \zeta)}{h}$$

Having a discrete PV relationship helps insure robust simulations.



[Ref: 5,6,7,20]



Can we make our discrete numerical model do this?

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0, \quad q = \frac{(f + \zeta)}{h}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{u} = -g\nabla h - \nabla K$$

Having a discrete PV relationship helps insure robust simulations.



[Ref: 5,6,7,20]



Can we make our discrete numerical model do this?

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$$\frac{\partial \mathbf{u}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{u} = -g\nabla h - \nabla K$$

$$\mathbf{k} \cdot \nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{u} = -g\nabla h - \nabla K \right]$$

Having a discrete PV relationship helps insure robust simulations.





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$$\mathbf{k} \cdot \nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{u} = -g\nabla h - \nabla K\right]$$

$$\frac{\partial hq}{\partial t} + \nabla \cdot (hq\mathbf{u}) = 0$$

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Having a discrete PV relationship helps insure robust simulations.





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$$\frac{\partial \mathbf{u}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{u} = -g\nabla h - \nabla K$$

$$\mathbf{k} \cdot \nabla \times \begin{bmatrix} \frac{\partial \mathbf{u}}{\partial t} + (f + \zeta)\mathbf{k} \times \mathbf{u} = -g\nabla h - \nabla K \end{bmatrix}$$

$$\frac{\partial hq}{\partial t} + \nabla \cdot (hq\mathbf{u}) = 0$$
Having a discrete PV relationship helps insure robust simulations.

$$\frac{Dq}{Dt} = 0$$
 Having a discrete PV relationship helps insure robust simulations.

[Ref: 5,6,7,20]



Summary for the Introduction to the Design of Dynamical Cores:

- I. The construction of a robust dynamical core is a necessary (but certainly not sufficient) condition to the accurate simulation of the climate system.
- 2. The design of a robust dynamical core requires a consideration of the general circulation, turbulence modeling and PDE analysis.
- 3. I doubt that there will ever by a single dynamical core suitable for all climate applications. Dynamical core design should consider the target application and make appropriate model choices based on that application.



End of Lecture #1.



Motivation for the construction of a multi-scale climate system model.

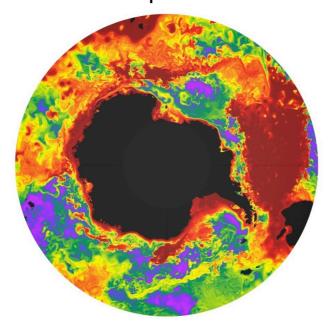


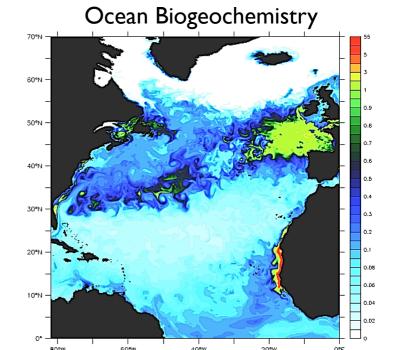
Examples of processes that are currently unresolved.

Cloud Processes

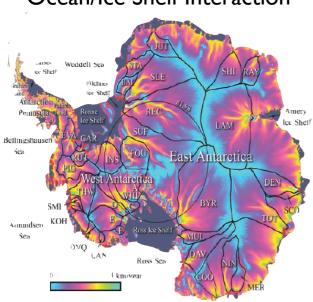


Ocean/Atmosphere Interaction





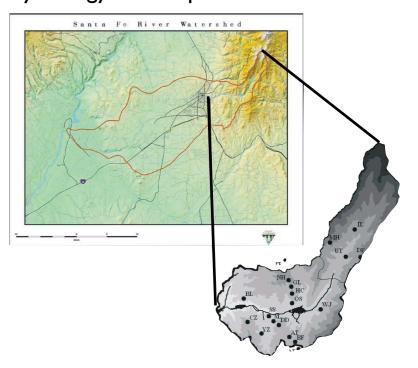
Ocean/Ice Shelf Interaction



Each of these examples demonstrate scale-sensitive processes that might impact the climate system in a fundamental and important way.

The length scale of these processes is O(km).

Hydrology in Complex Terrain

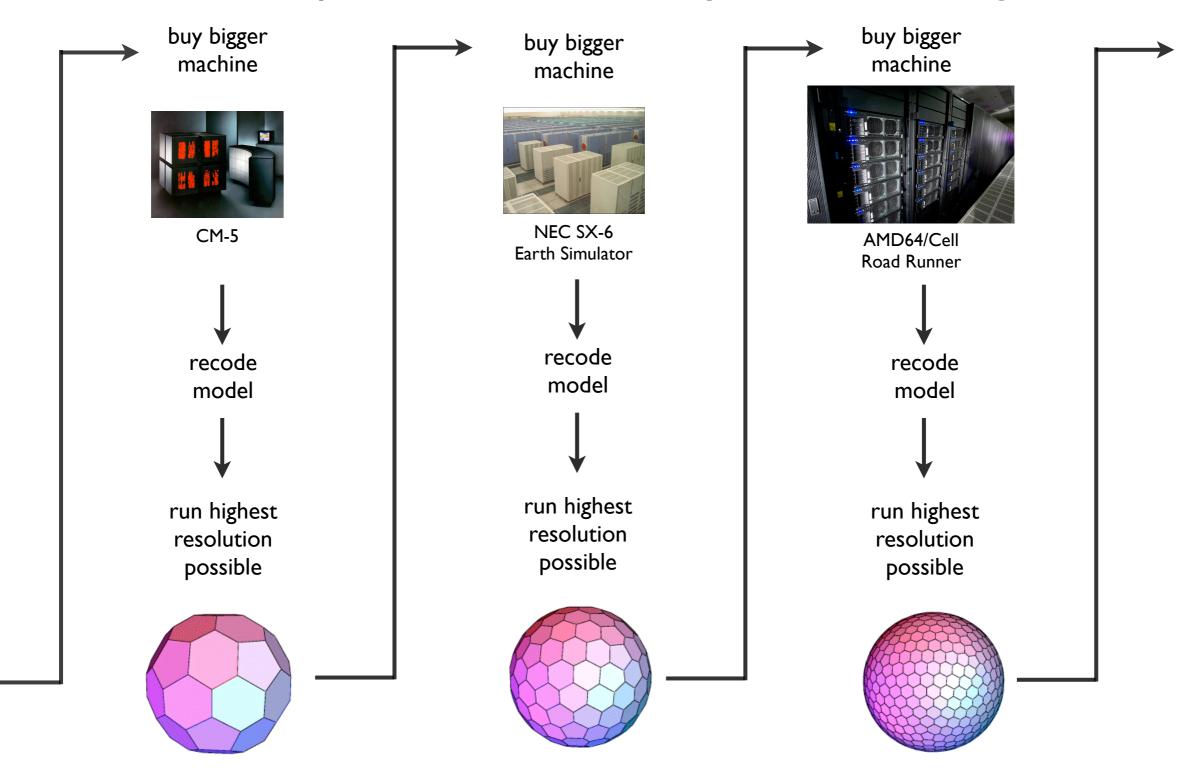


[Ref: 21]





Modus Operandi of Climate System Modeling



Bigger, faster computers are fantastic. Recoding for those machine is simply the cost of progress. The question is how do we distribute those resources?





How do we get there with the current paradigm?

A quick back-of-the-envelope analysis should be very concerning to mid-career scientists:

IPCC resolution ~ 100 km

Target resolution ~ I km

Ratio of where we are to where we want to be $\sim 2^7$

Increase in computational resource required $\sim 8^7 \sim 2e6$

Time to reach target assuming a doubling in resources every 18 months ~ 22 years

Looking at the massive effort required to increase the resolution climate models, 22 years is not unrealistic. Maybe it is 15 years, or maybe it it 25 years. Either way, it is a long time.



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Looking at the massive effort required to increase the resolution climate models, 22 years is not unrealistic. Maybe it is 15 years, or maybe it it 25 years. Either way, it is a long time.

The climate modeling community would benefit from an <u>additional</u> approach that allows <u>some</u> of these processes to be resolved at <u>some</u> locations.

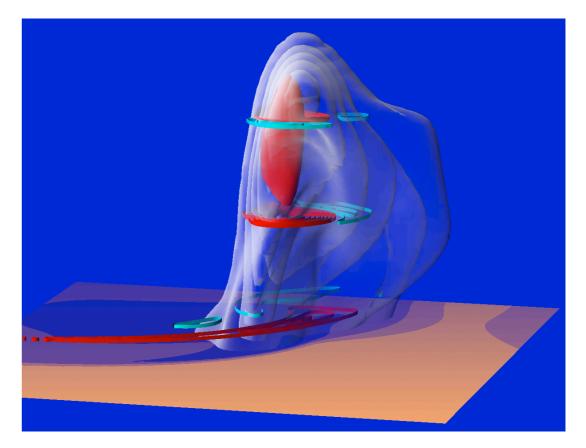
We could then consider the notion of approaching the target from additional directions, for example by expanding the areas over which these fine-scale processes are simulated.





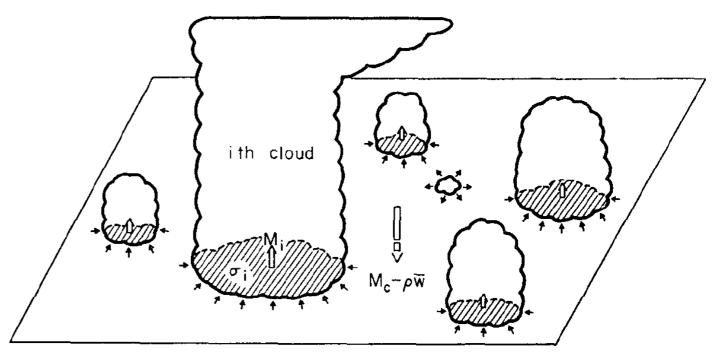
Clouds in Global Atmosphere Models





From top left moving clockwise:

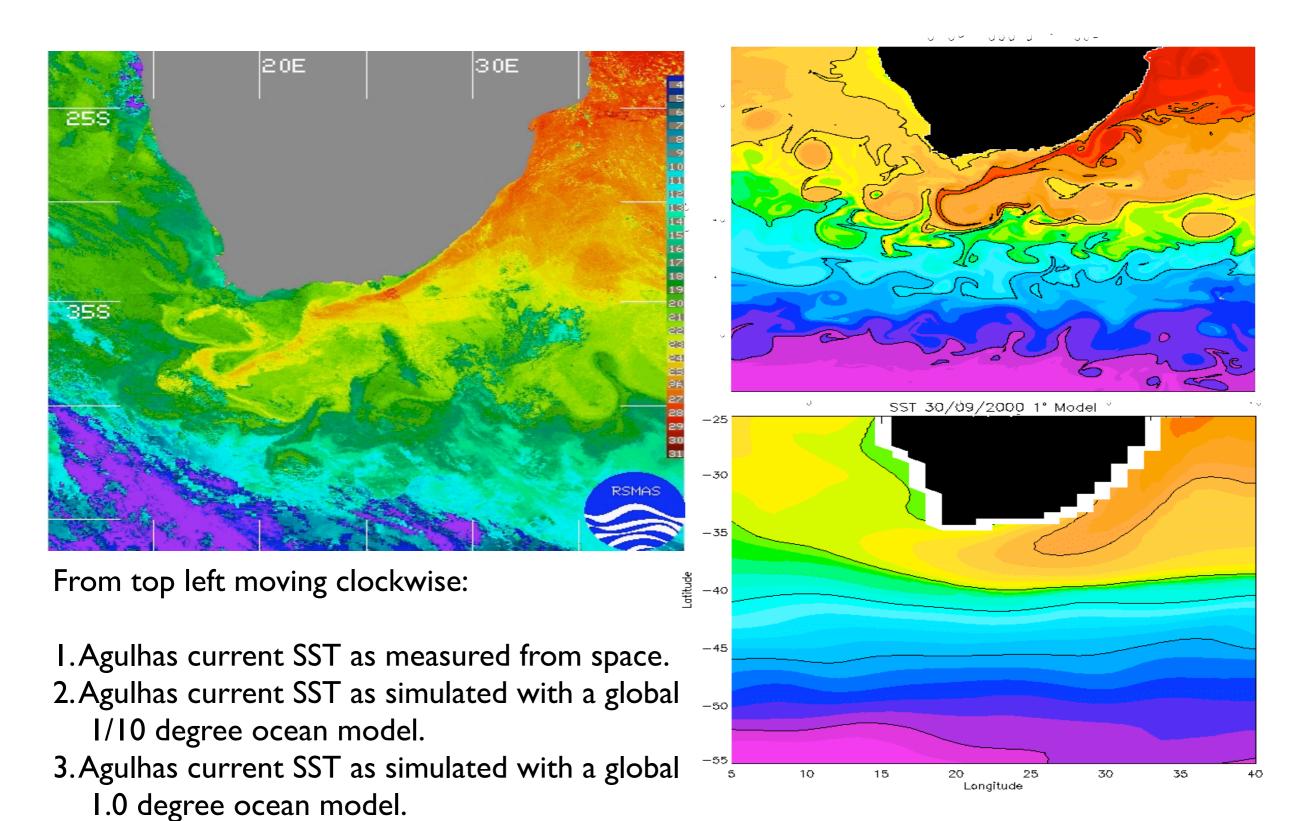
- I.A supercell as observed
- 2. A supercell simulated with a non-hydrostatic model.
- 3. A parameterization of clouds that depend solely on the large-scale environment.







Eddies in Global Ocean Models

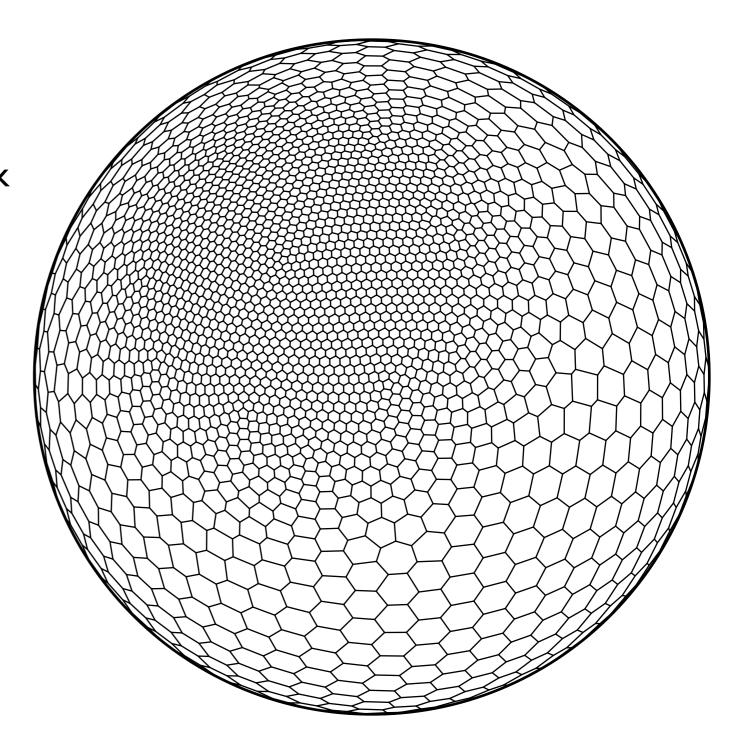


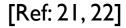


A multi-resolution approach based on Spherical Centroidal Voronoi Tessellations

Underlying principles of this approach:

- I. This is a global modeling framework
- 2. The approach allows us to "paint" the sphere with regions enhanced resolution.
- 3. The resulting mesh is conforming (i.e. no hanging nodes) and is a Voronoi diagram.







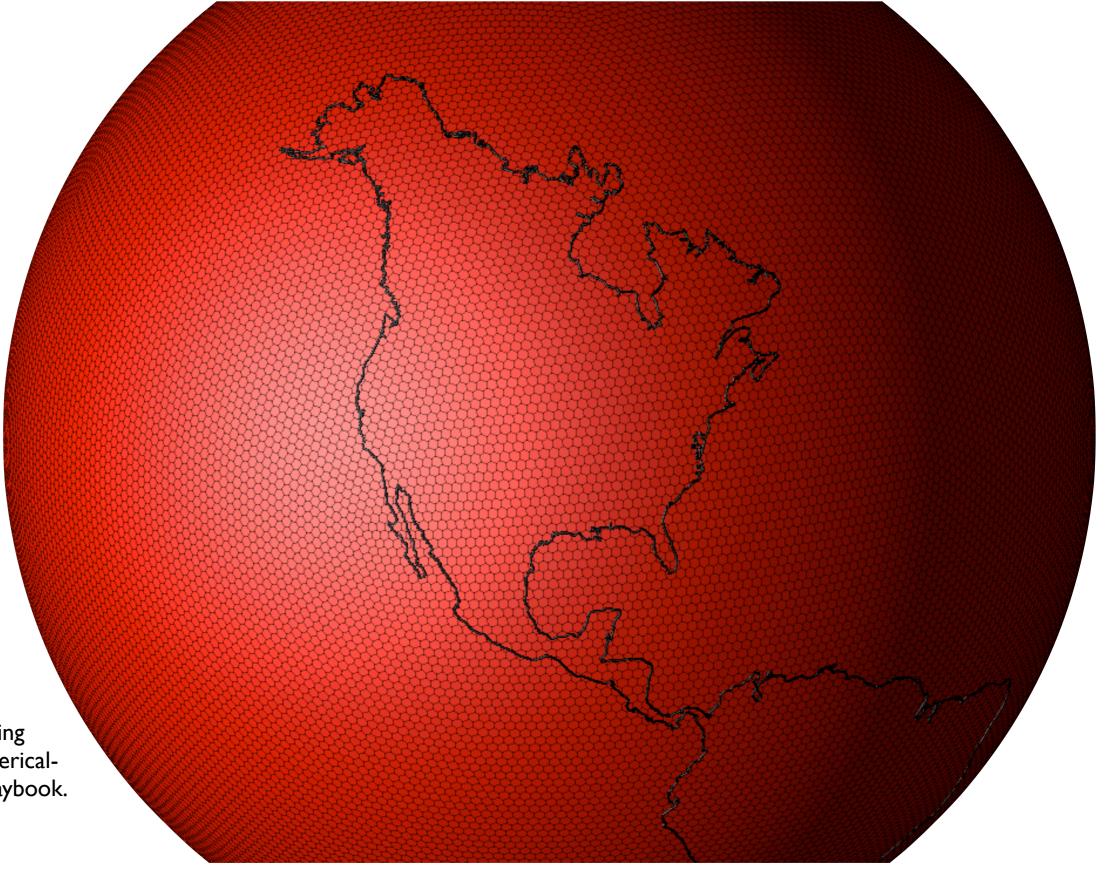
Variable resolution meshes.



In order to mitigate our lack of scale-aware physical parameterizations, we are focused on a nested approach.

The approach is equally valid for atmosphere, ocean, and ice modeling.

This has the added benefit of allowing multiple time-steps. A short time step in the nested domain and a longer time step in the coarse domain.



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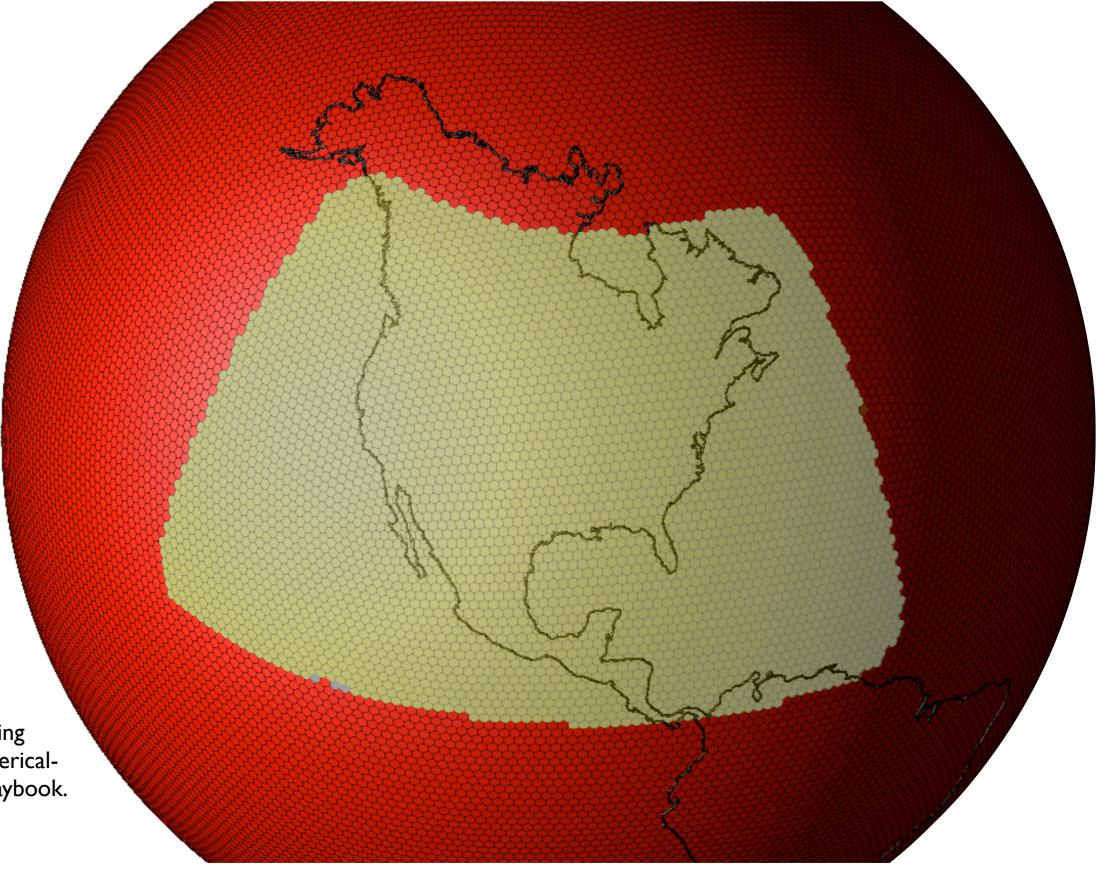




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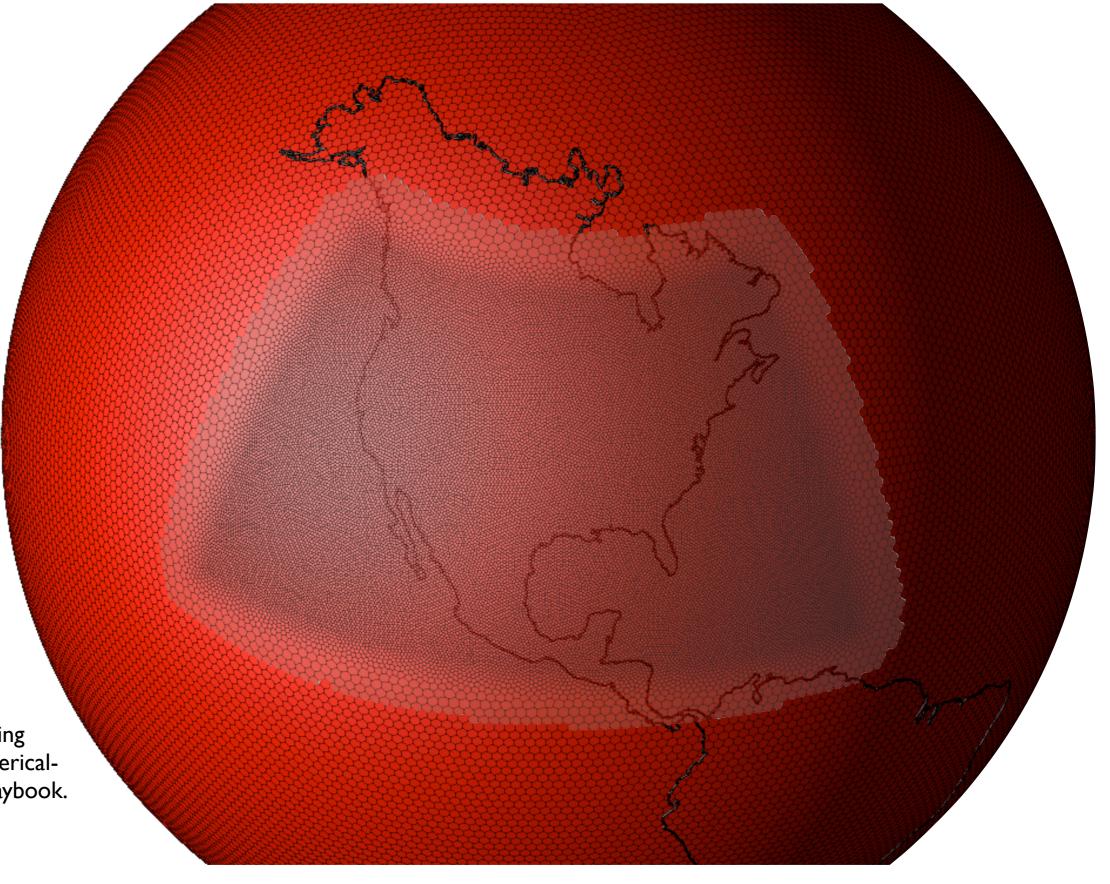
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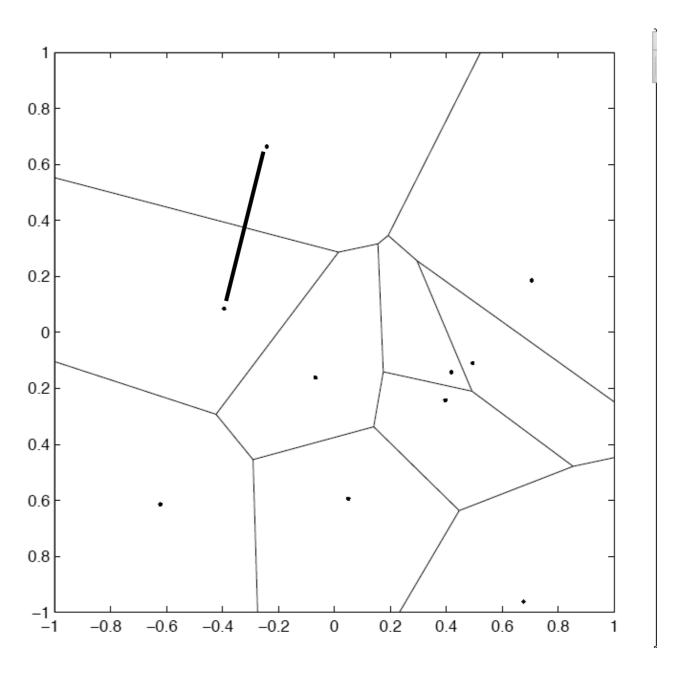


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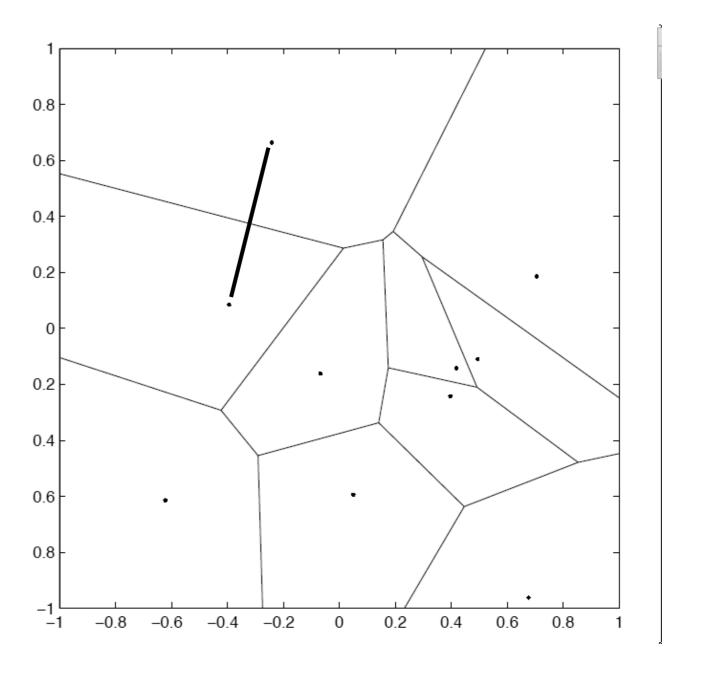








Given a region, S, and a set of generators, z_i ...

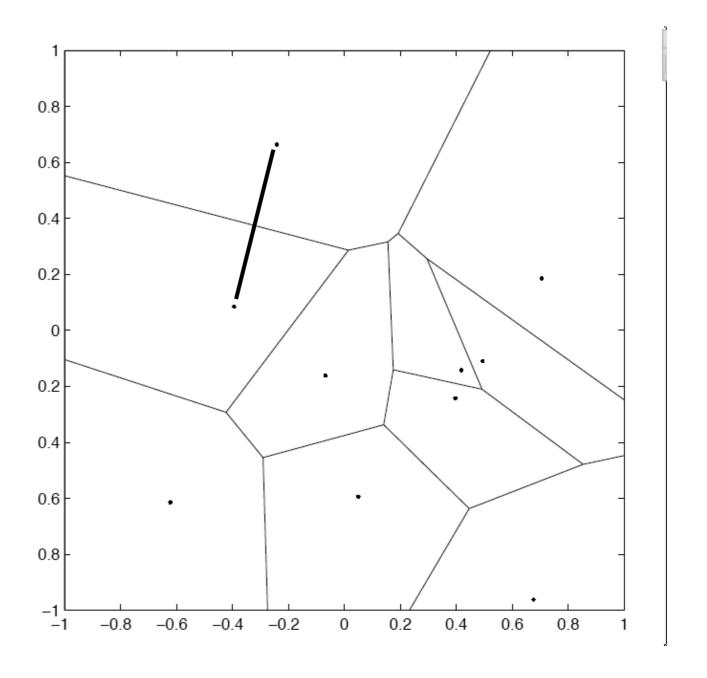






Given a region, S, and a set of generators, z_i ...

The Voronoi region, V_i , for each z_i is the set of all points closer to z_i than z_j for j not equal to i.





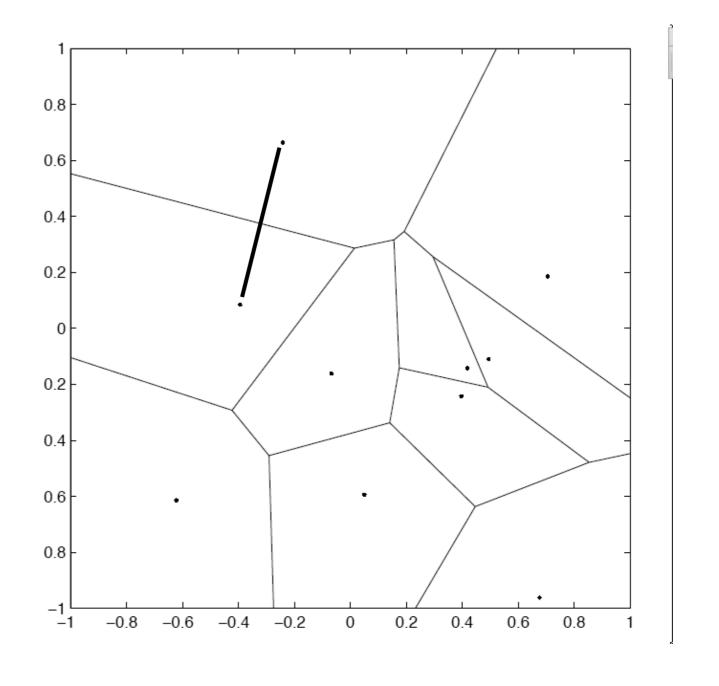


Given a region, S, and a set of generators, z_i ...

The Voronoi region, V_i , for each z_i is the set of all points closer to z_i than z_j for j not equal to i.

We are guaranteed that each edge (aka face) is shared by exactly two generators.

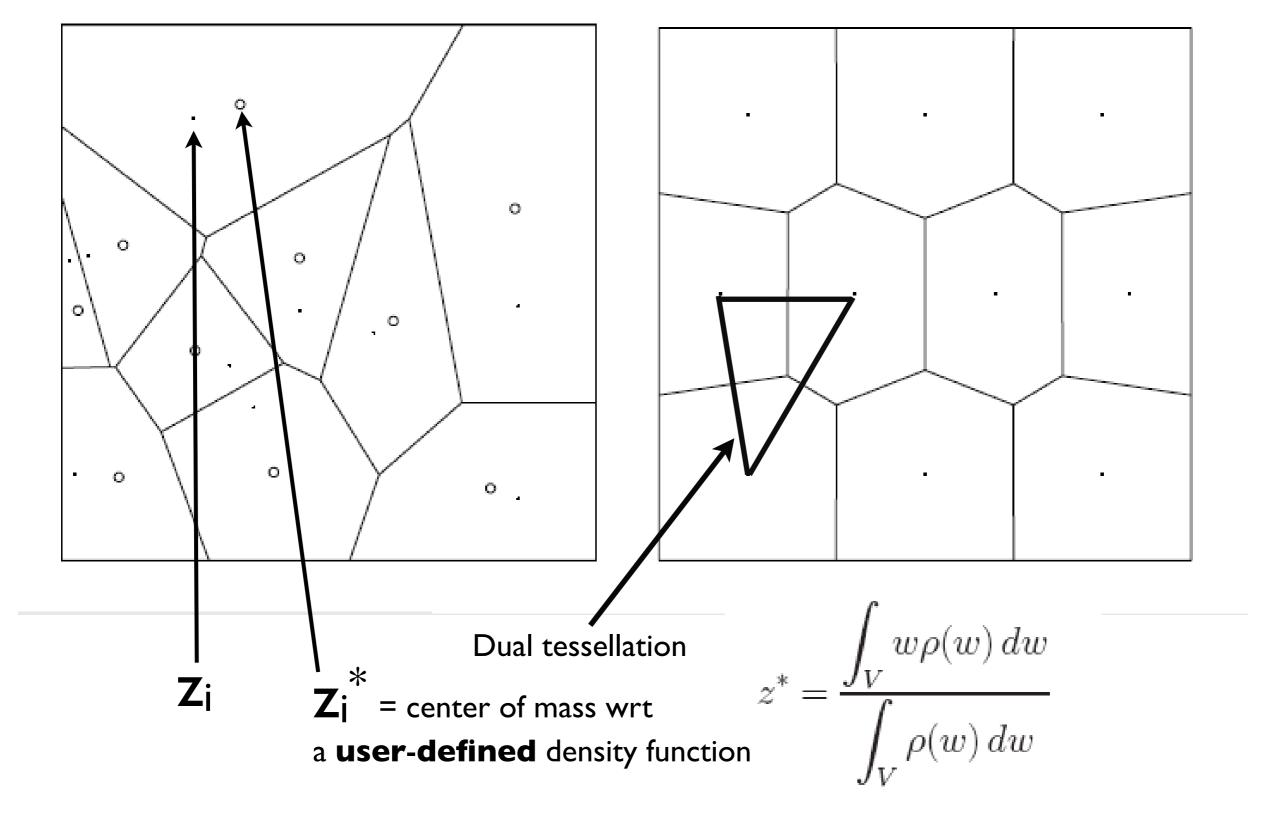
We are guaranteed that the line connecting generators is orthogonal to the shared edge and is bisected by that edge.







Definition of a Centroidal Voronoi Tessellation



[Ref: 24]

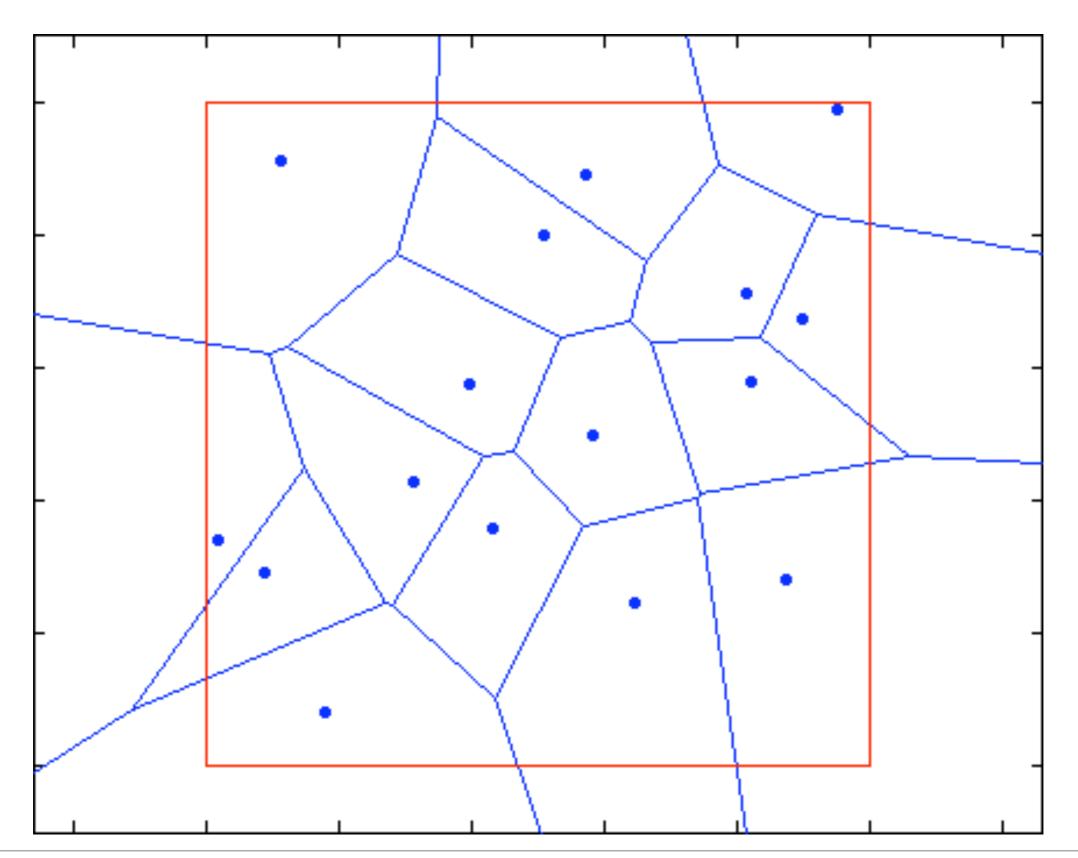


Centroidal Voronoi Diagrams: Voronoi diagram where generator is also center of mass.

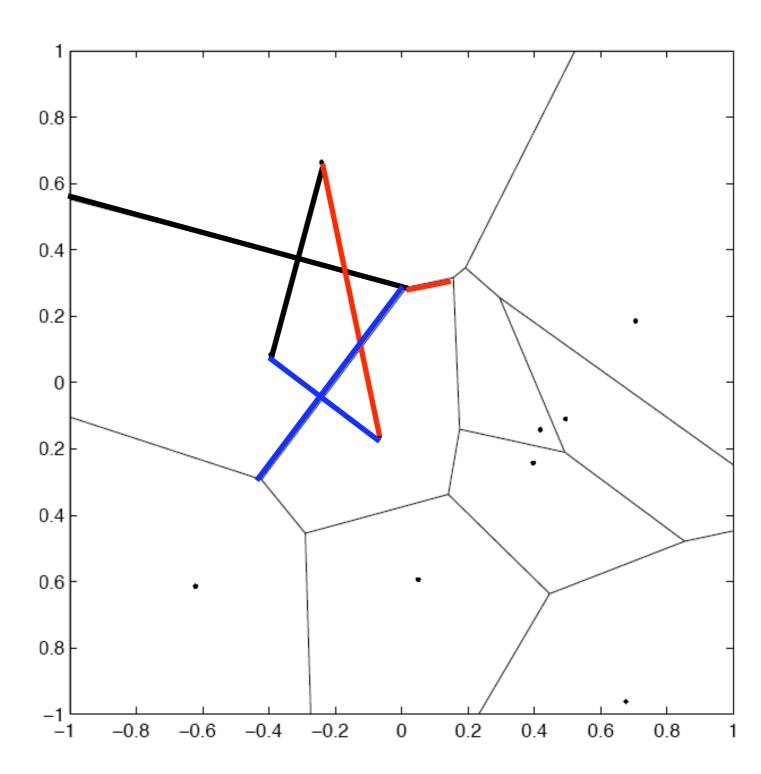




Centroidal Voronoi Diagrams: Voronoi diagram where generator is also center of mass.



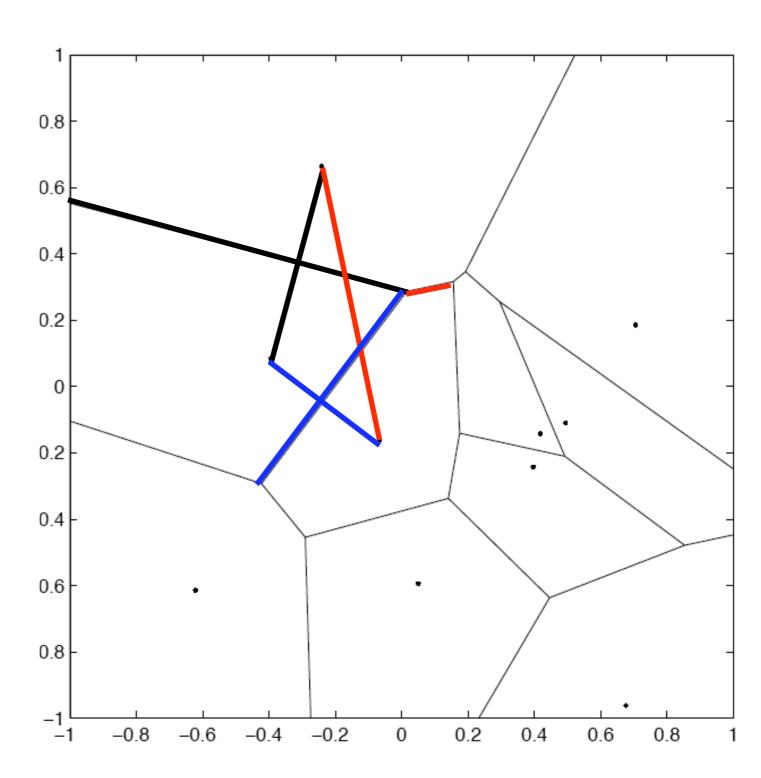








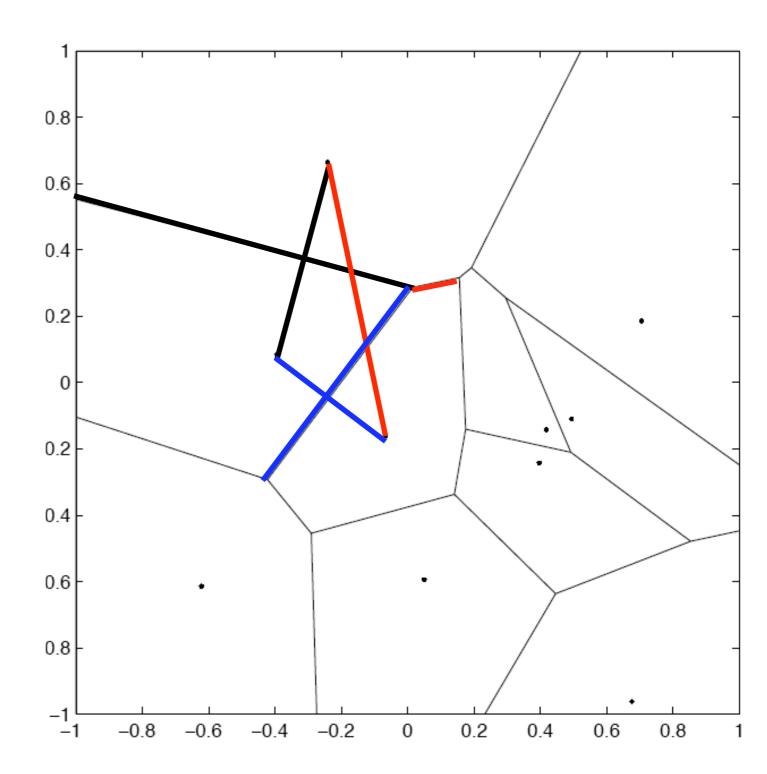
Each cell edge of the Voronoi diagram is uniquely associated with one cell edge of the Delaunay triangulation.





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Each pair of edges are orthogonal, (but do not necessarily intersect!)

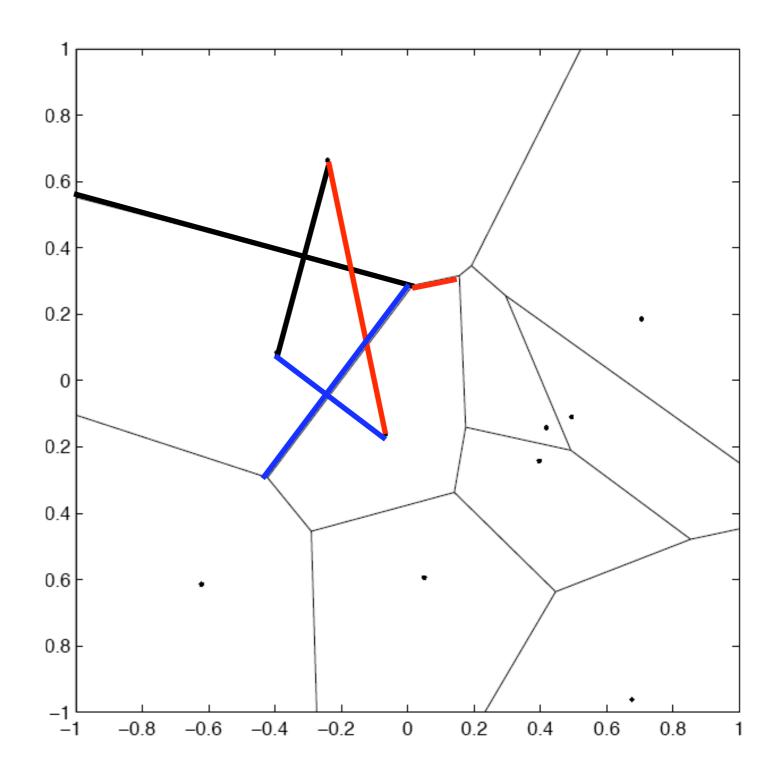




Each cell edge of the Voronoi diagram is uniquely associated with one cell edge of the Delaunay triangulation.

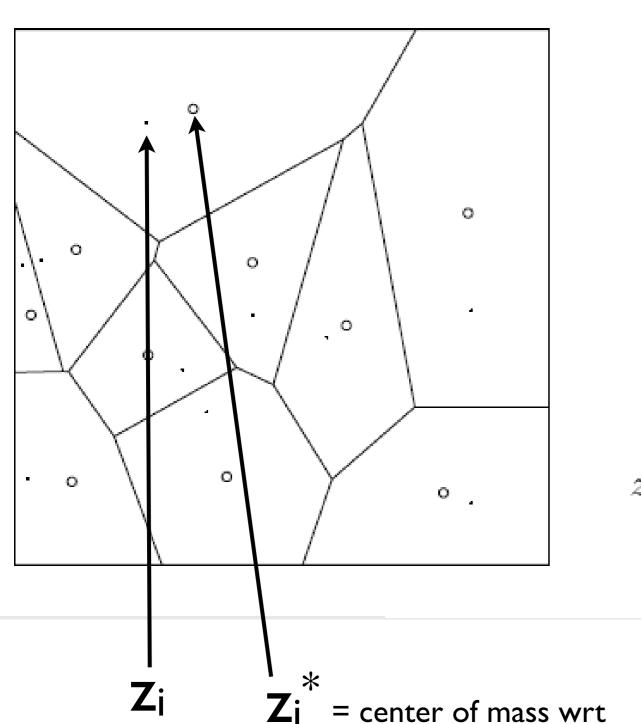
Each pair of edges are orthogonal, (but do not necessarily intersect!)

If the pair of edges do intersect (or if the lines segments are extended to point where they intersect), then the intersection point will bisect the line segment connecting generators.





Methods for determining the CVT: Deterministic



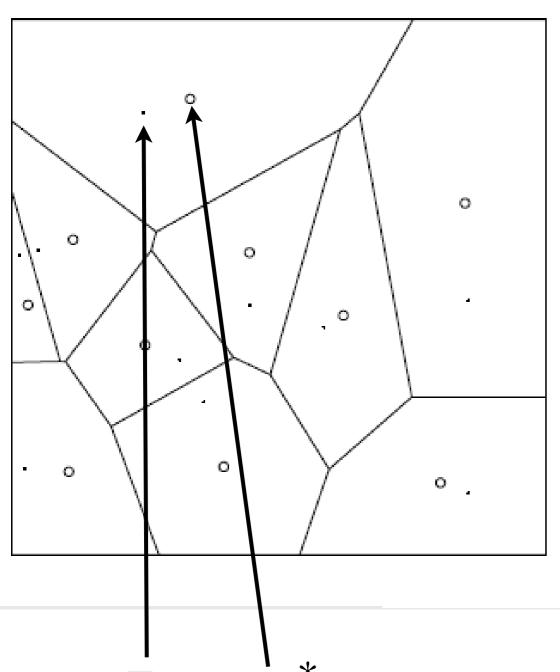
$$z^* = \frac{\int_V w \rho(w) \, dw}{\int_V \rho(w) \, dw}$$

Z_i = center of mass wrt
a **user-defined** density function





Methods for determining the CVT: Deterministic



Lloyd's Algorithm

- 1) Given a set of generators, draw Voronoi diagram
- 2) Find center of mass of each cell (via numerical integration)
- 3) Measure error (distance between generators and center of mass)
- 4) Move generators to center of mass
- 5) Error too big? If yes, to go 1).

$$z^* = \frac{\int_V w \rho(w) \, dw}{\int_V \rho(w) \, dw}$$

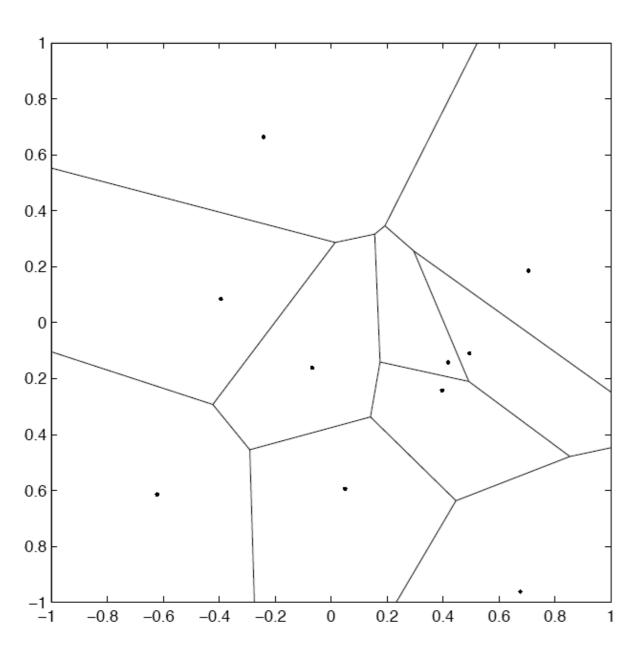
Z_i = center of mass wrt
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[Ref: 24, 25, 26]



Methods for determining the CVT: Statistical

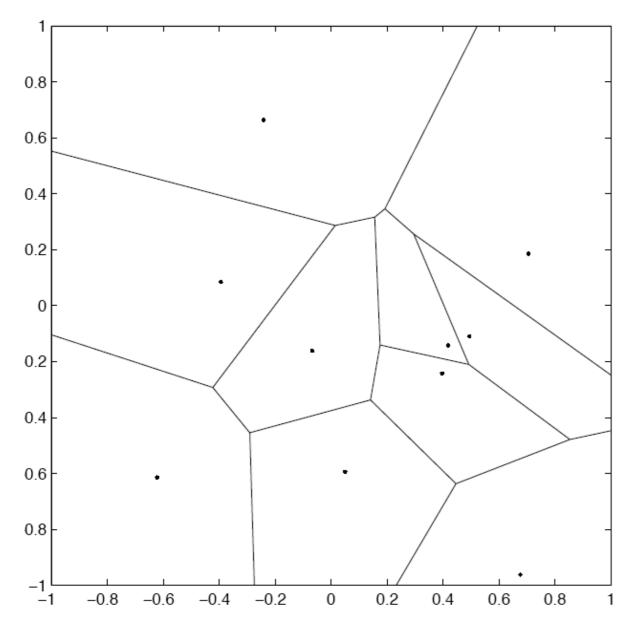






[Ref: 24, 27]

Methods for determining the CVT: Statistical



Statistical Sampling Algorithm

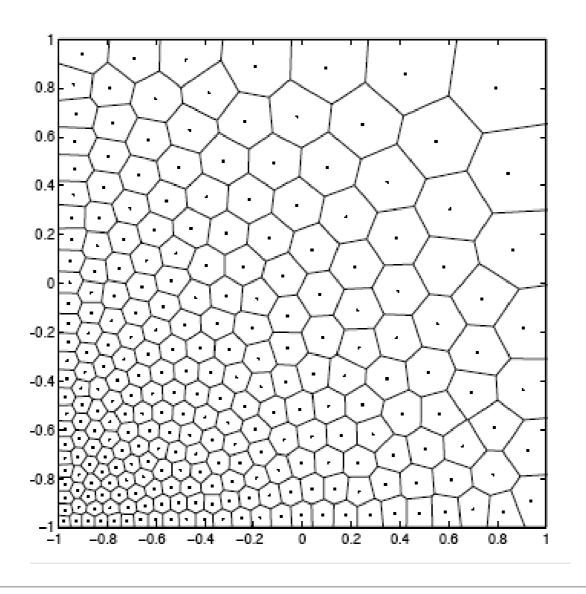
- I) Randomly sample a point, X, in the domain.
- 2) Generate a random number, R, between rho(min) and rho(max)
- 3) Discard point if R<rho(X)
- 4) Assign sample point to closest generator
- 5) Return to 1) until N points have been retained
- 6) Move generator to arithmetic mean of associated sample points.
- 7) Measure error (distance of generators movement)
- 6) Error too big? If yes, to go 1).





Gersho's conjecture

Pick some CVT density field (with minimal constraints on smoothness). Gersho's conjecture (now proven in 2D) tells us that as we add generators all of the cells evolve toward perfect hexagons. By extension, the dual Delaunay triangulation evolves toward equilateral triangles.



[Ref: 28, 29]



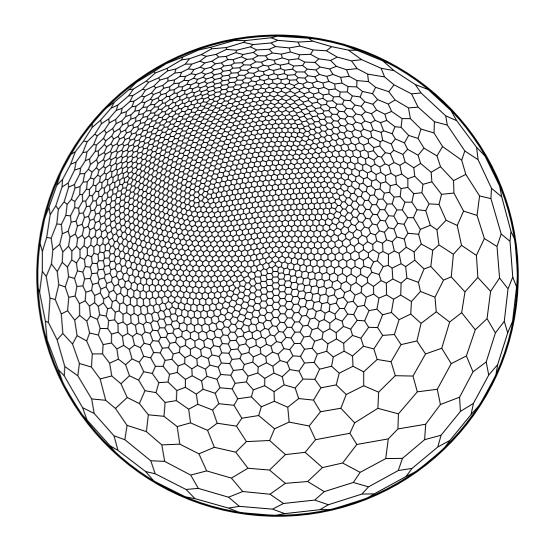
As far as mesh generation goes, SCVTs have a strong mathematical foundation

$$\frac{h(\mathbf{x_i})}{h(\mathbf{x_j})} \approx \left(\frac{\rho(\mathbf{x_j})}{\rho(\mathbf{x_i})}\right)^{d+2}$$

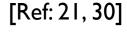
d: dimension of space to tessellate

 $h(\mathbf{x})$: nominal grid spacing

 $ho(\mathbf{x})$: user-defined density function







Defining a family of density functions

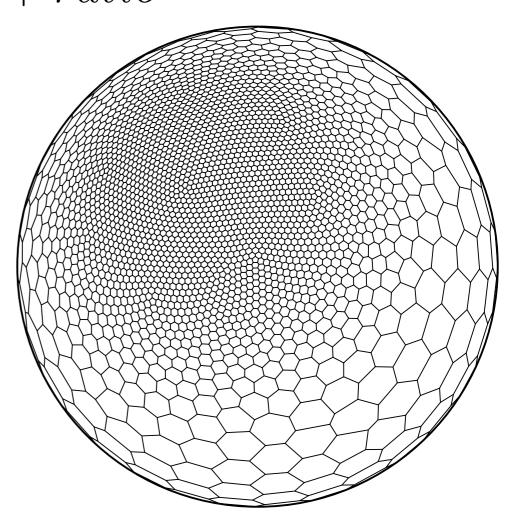
$$\rho(\mathbf{x}_i) = \frac{1}{2\beta} \left(\tanh \left(\frac{size - r}{width} \right) + 1 \right) + ratio$$

size : size of high-res region

width: width of transition zone

$$ratio = \left[\frac{h(\mathbf{x}_{low})}{h(\mathbf{x}_{high})}\right]^{\frac{1}{4}}$$

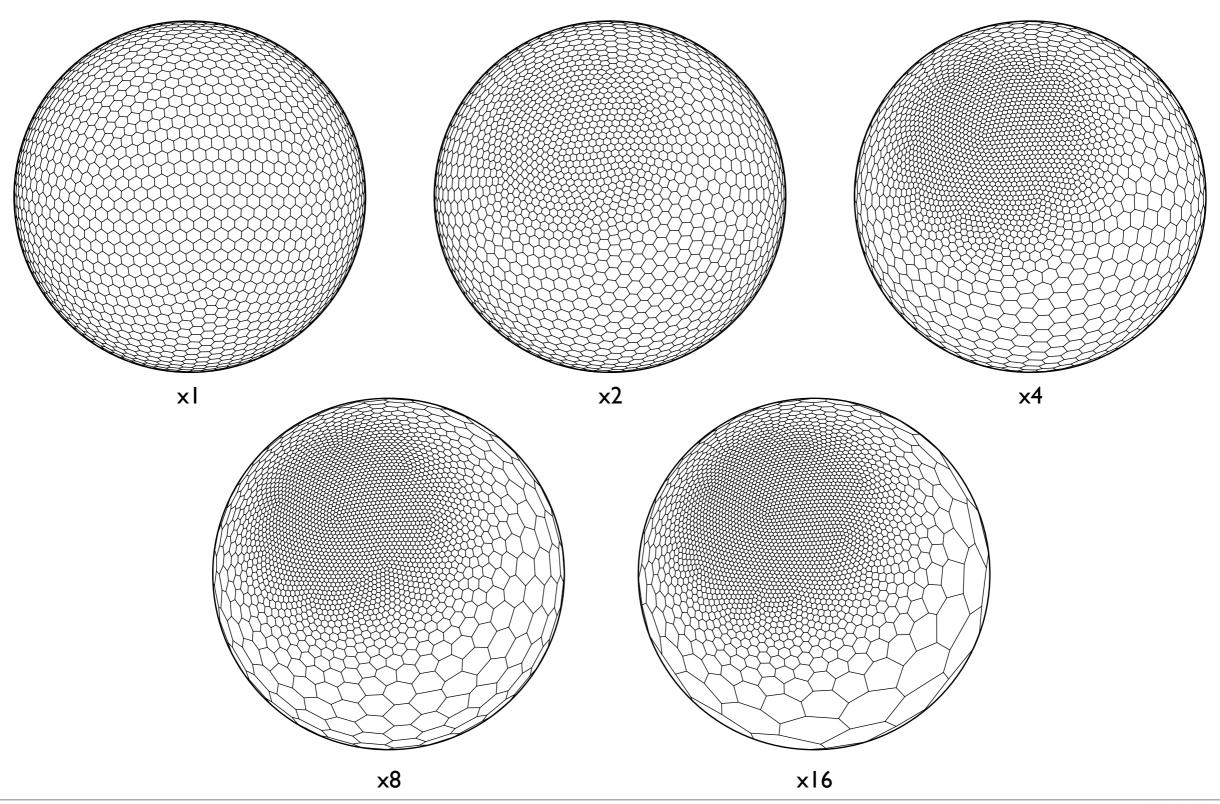
r: distance from center of high-res region



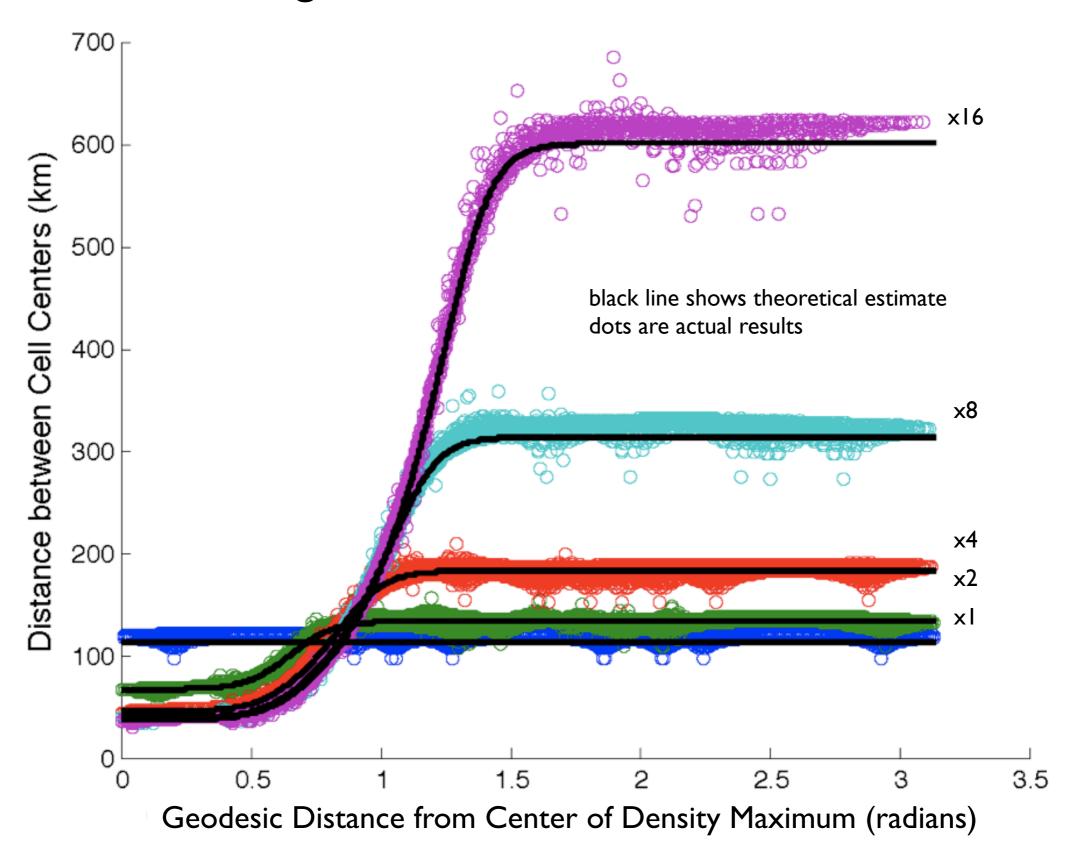
The density function is a simple three parameter system from which only the impact of the ratio parameter will be explored.



Spherical Centroidal Voronoi Tessellations (SCVTs): various ways to distribute 2562 nodes on the sphere.



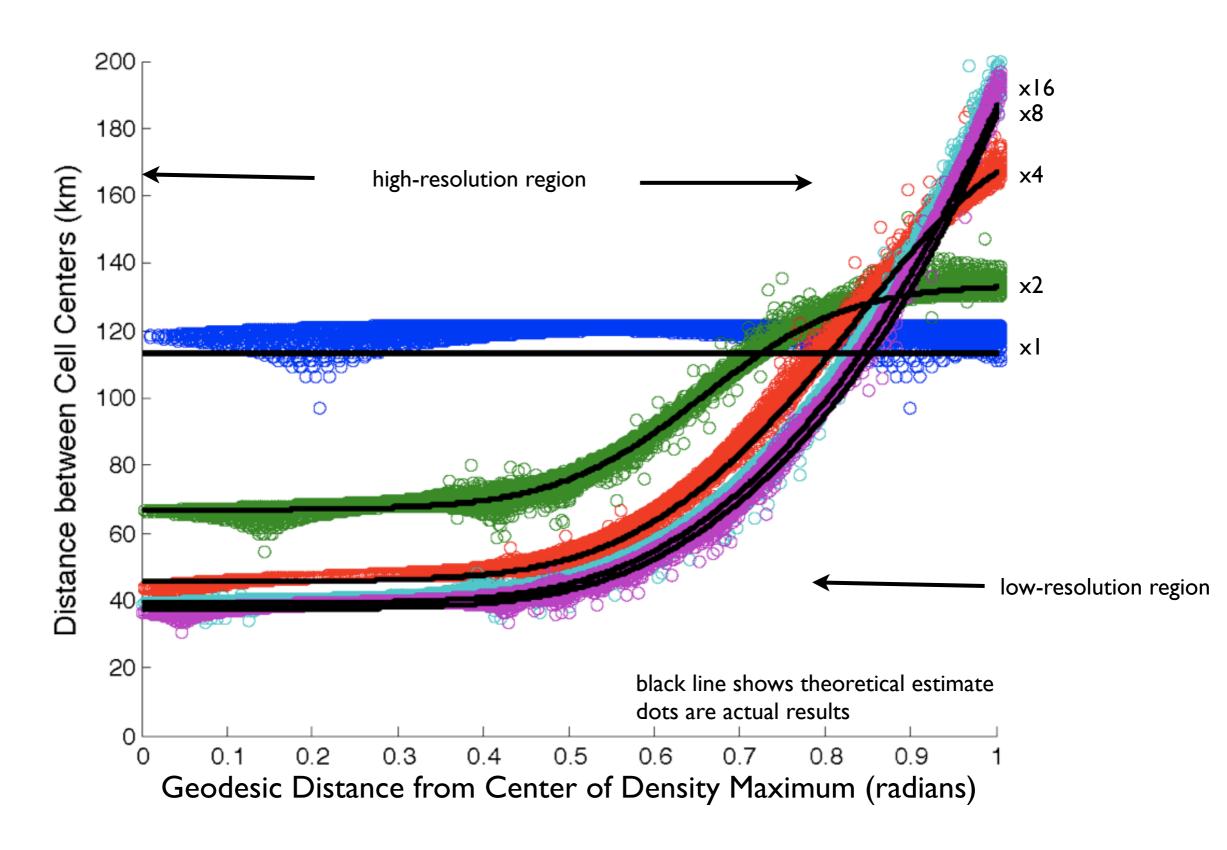
Some grid metrics with 40962 nodes





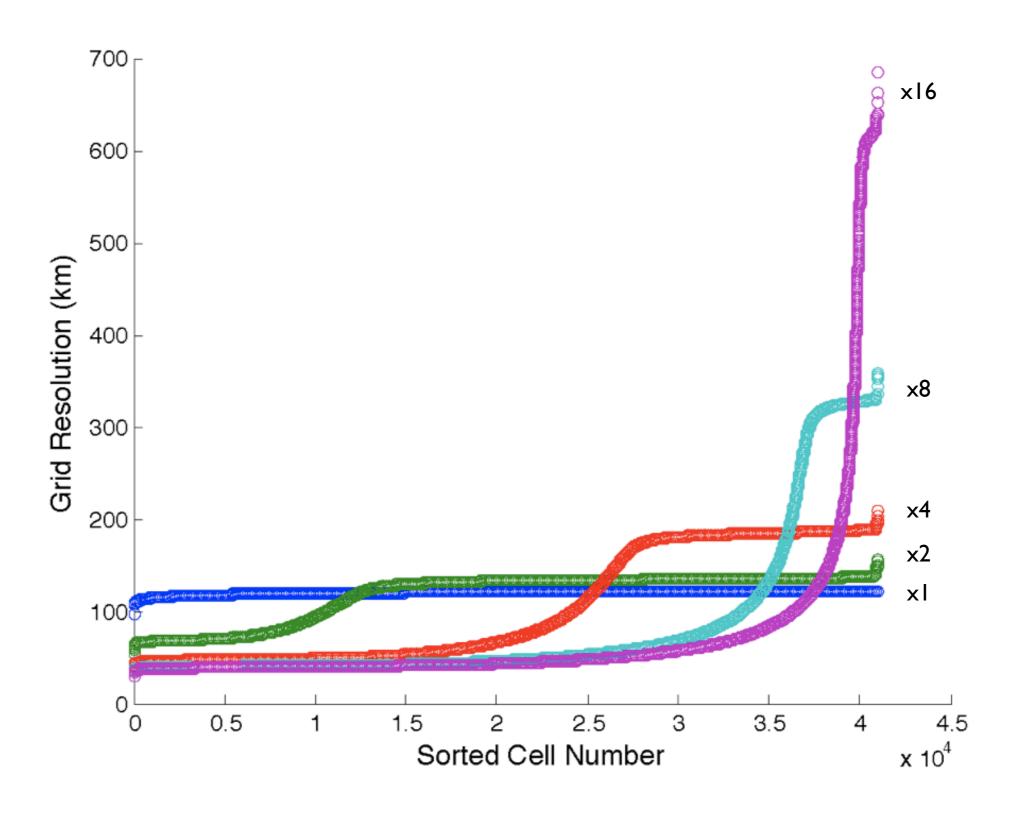


Some grid metrics with 40962 nodes



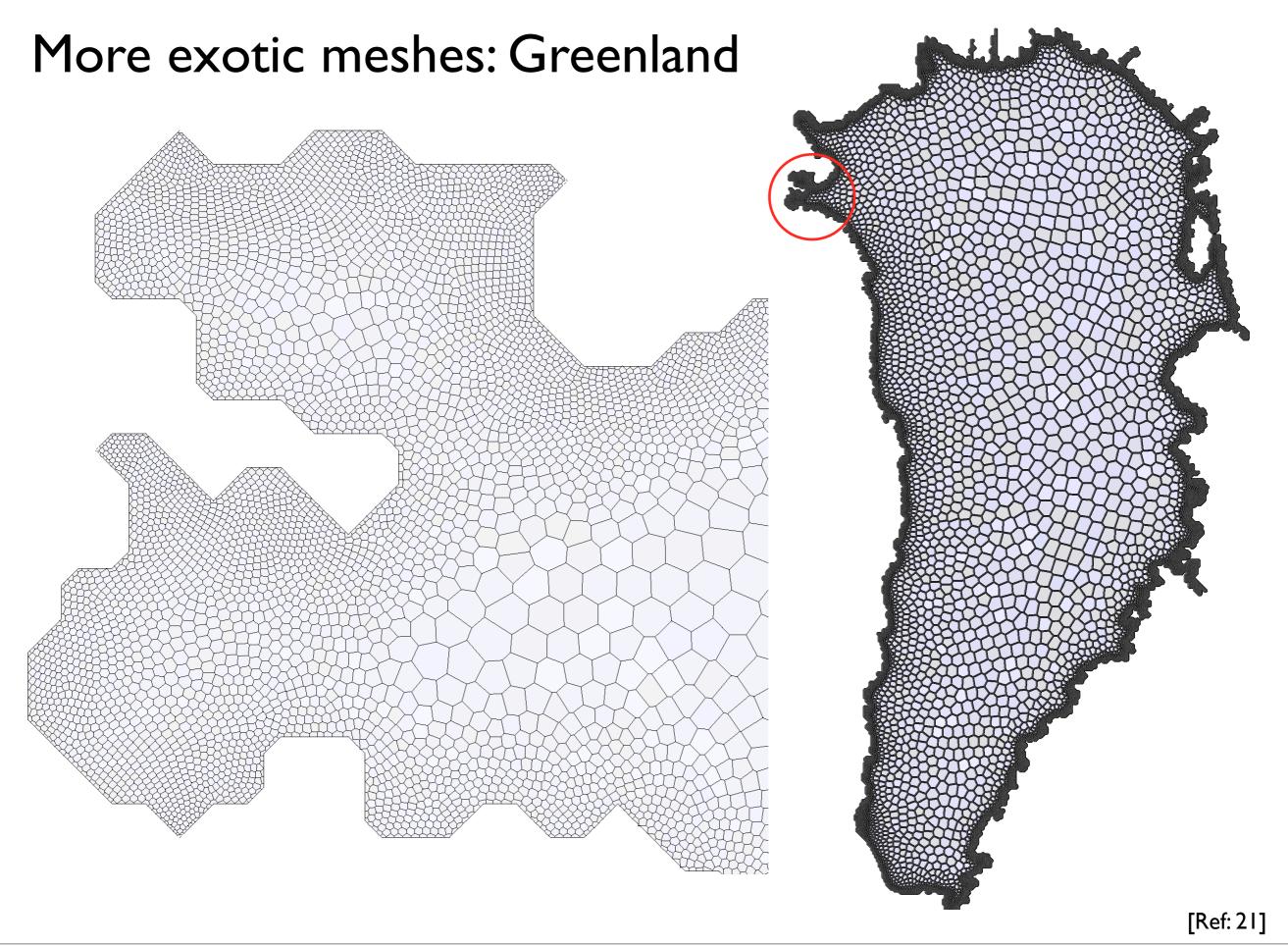


Some grid metrics with 40962 nodes





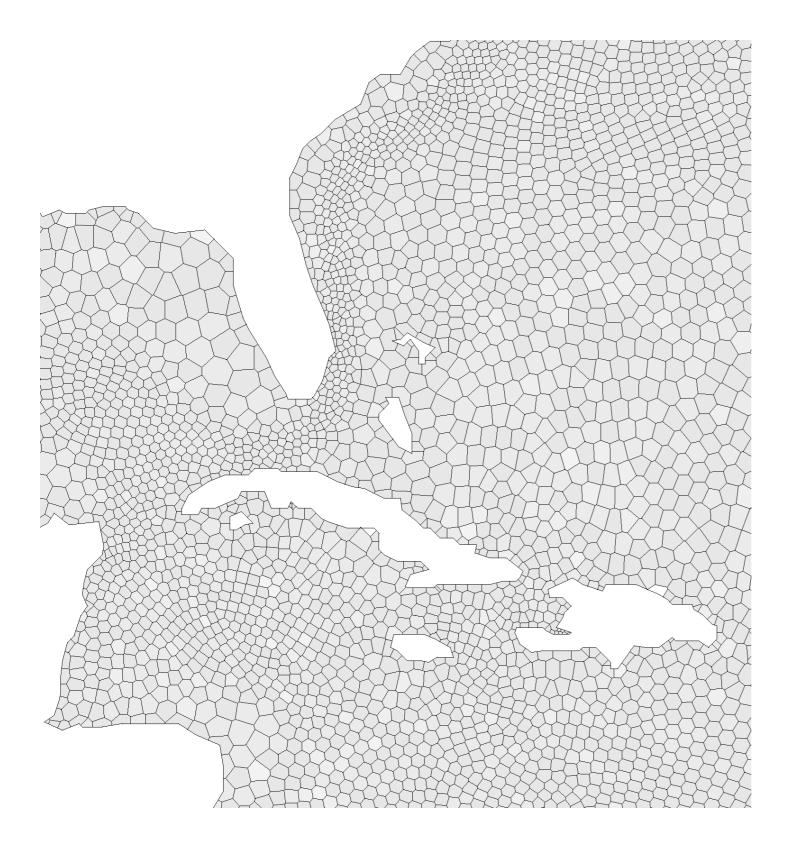








More exotic meshes: North Atlantic



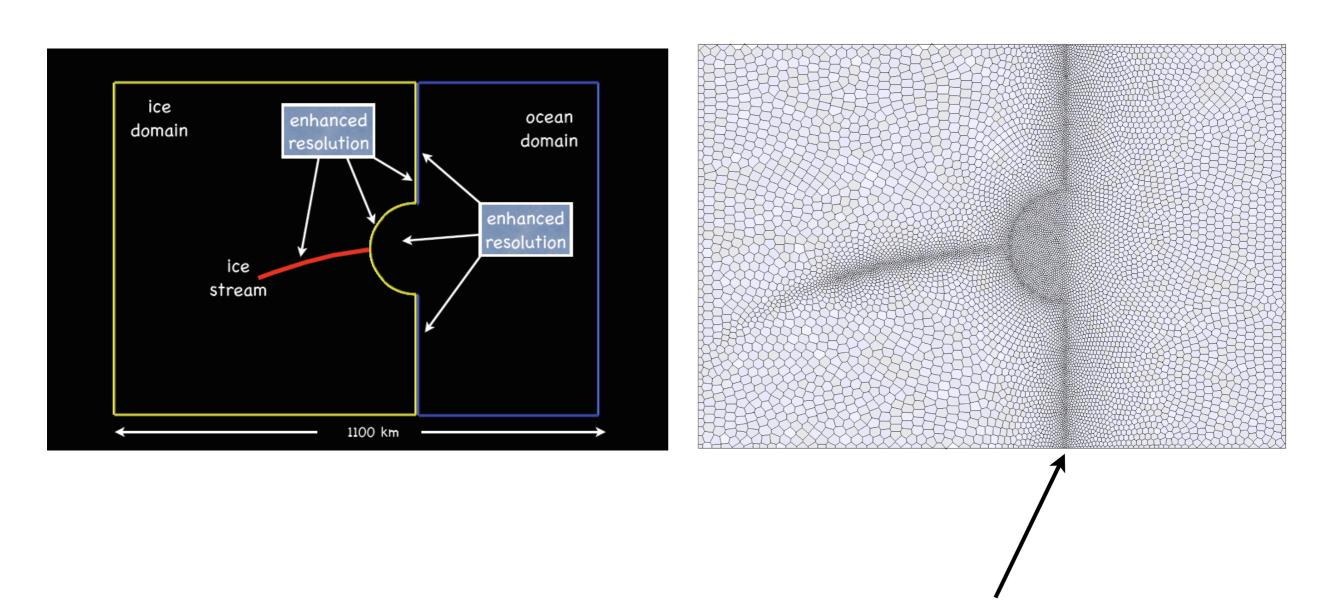
Mesh of the North Atlantic:
Density function chosen to be
proportional to ocean kinetic
energy so that resolution is placed
in strongly-eddying regions.



[Ref: 21]



More exotic meshes: Ocean-Land Ice coupled system.



The mesh "continuous" across the ocean/ice interface.

[Ref: 21]



Summary of the mesh generation technology:

- I. The Voronoi diagram has a solid foundation in applied mathematics. This solid foundation will be exploited in the following section on numerical methods.
- 2. The SCVT approach offers the capability to produce high-quality, variable resolution meshes based on a user-defined mesh density function.
- 3. These variable resolution meshes can be as simple or as complicated as the user chooses.



Design of a finite-volume solver for variable resolution meshes.





Equation Set

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + q(h\mathbf{u}^{\perp}) = -g\nabla (h + h_s) - \nabla K$$

$$\eta = \nabla \times \mathbf{u} + f$$

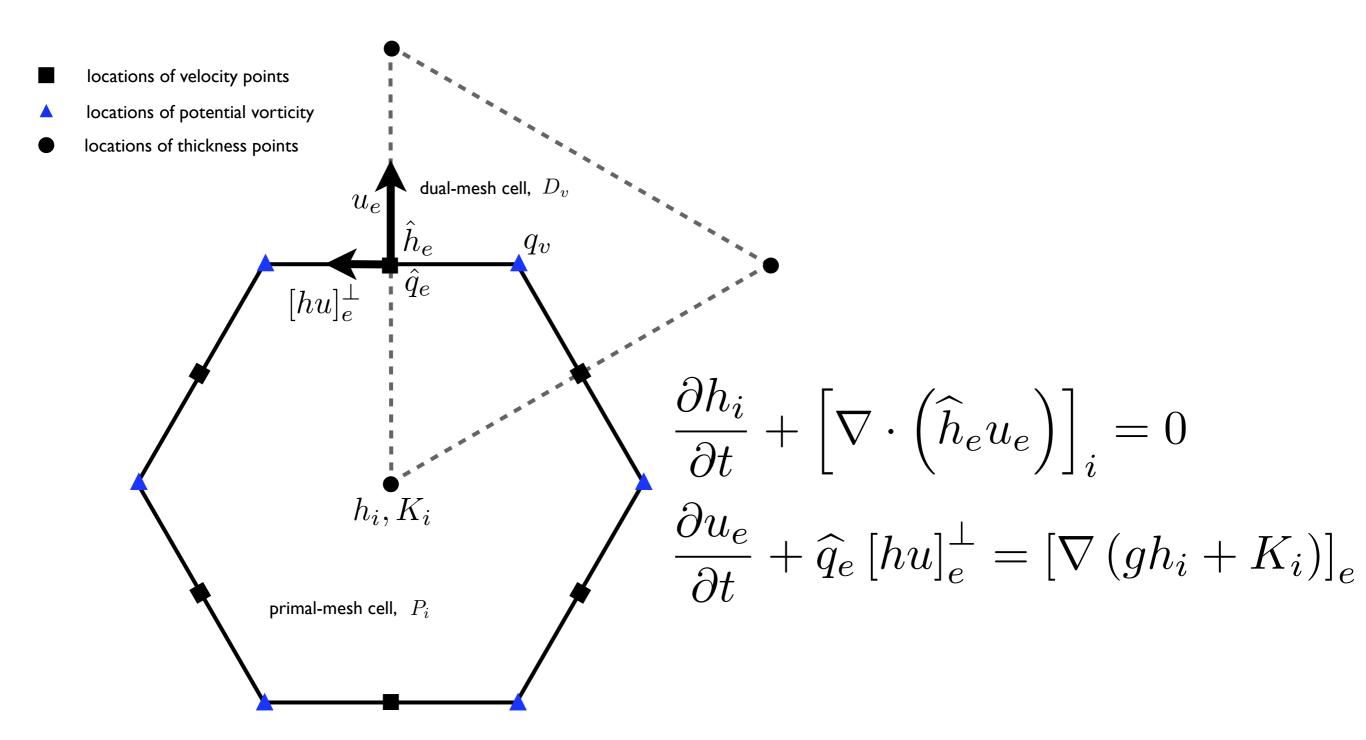
definition:

$$\mathbf{u}^\perp = \mathbf{k} imes \mathbf{u}$$

$$q = \frac{\eta}{h}$$



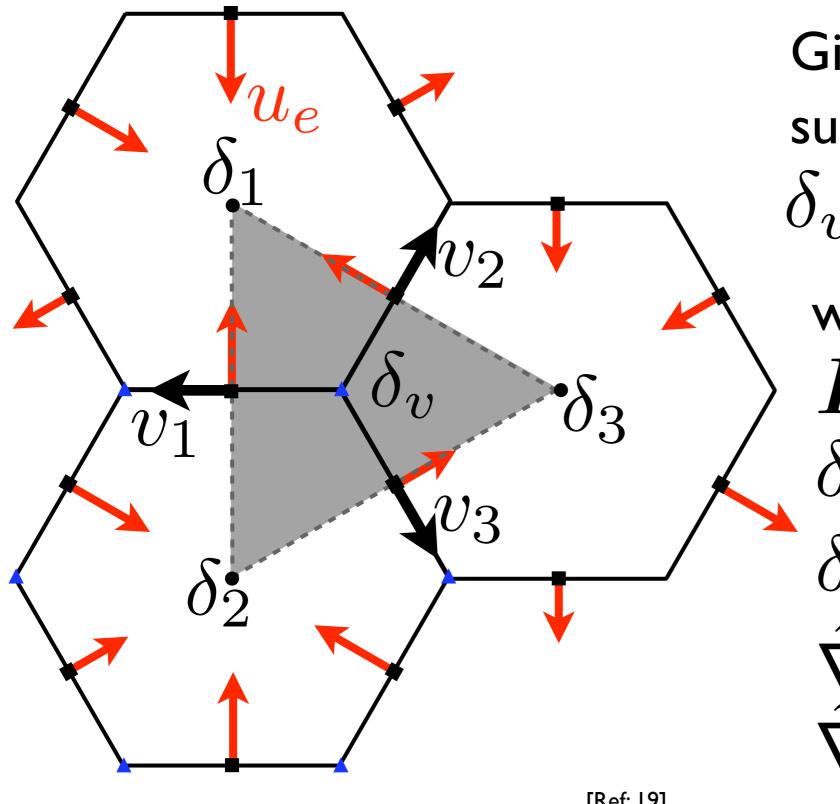
Defining the discrete system







The key to developing our robust multi-resolution method was solving this math problem



Given u_e , find $v_{1,2,3}$ such that

$$\delta_v = I(\delta_1, \delta_2, \delta_3)$$

where

T is a convex interpolant

$$\delta_v = \tilde{\nabla}_d(v_{1,2,3})$$

$$\delta_i = \tilde{\nabla}_p(u_e)$$

discrete div on dual p discrete div on prime

[Ref: 19]



Relationship between nonlinear Coriolis force and potential vorticity flux

$$\mathbf{k} \cdot \nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} + q(h\mathbf{u}^{\perp}) = -g\nabla (h + h_s) - \nabla K \right]$$

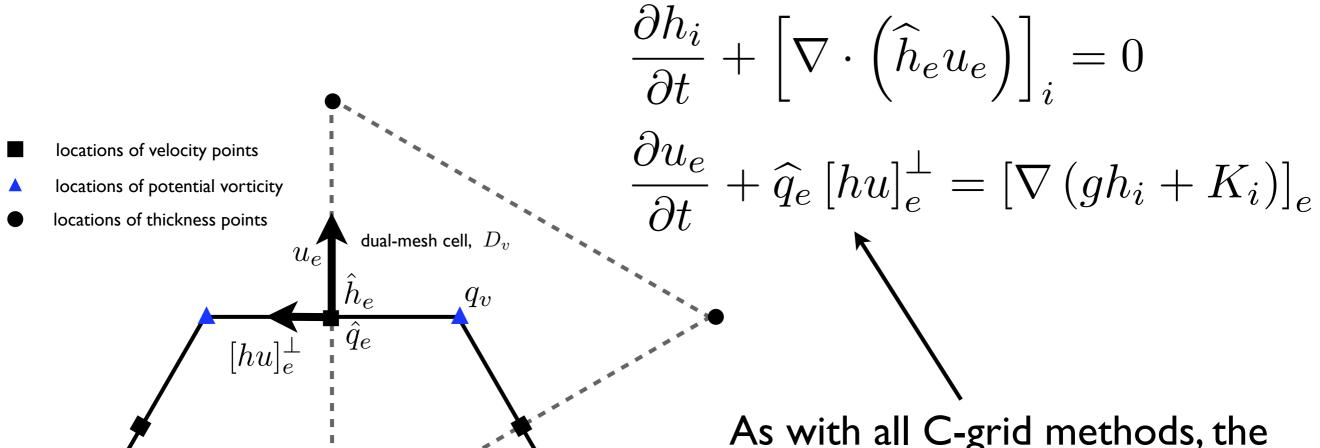
$$\frac{\partial \eta}{\partial t} + \mathbf{k} \cdot \nabla \times \left[\eta \mathbf{u}^{\perp} \right] = 0$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[\eta \mathbf{u} \right] = 0$$
potential vorticity flux
$$\frac{\partial (hq)}{\partial t} + \nabla \cdot [hq\mathbf{u}] = 0$$

The nonlinear Coriolis force IS the PV flux in the direction perpendicular to the velocity.

[Ref: 13, 19]

Construction of a multi-resolution finite volume method



As with all C-grid methods, the biggest challenge is how to reconstruct the nonlinear Coriolis force. In this system, ALL vorticity dynamics are contained in this terms.

- I) How to compute $[hu]_e^{\perp}$?
- 2) How to compute \widehat{q} ?

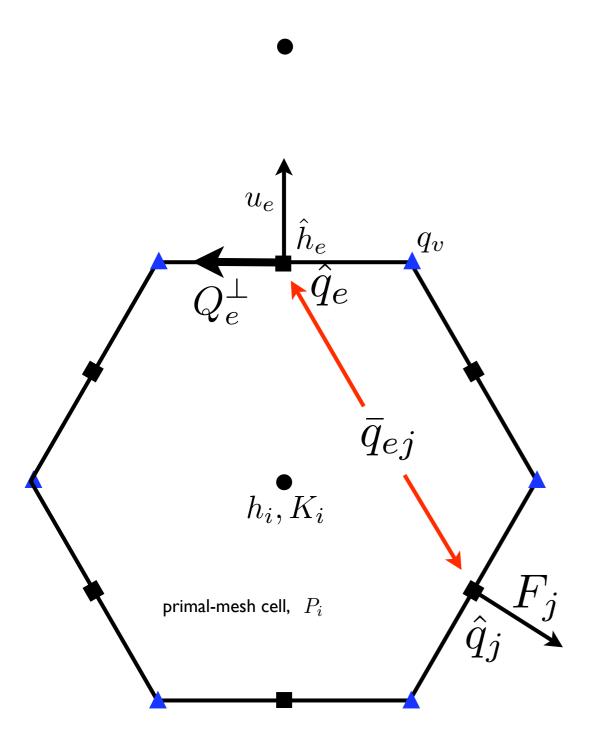
[Ref: 13, 19]

 h_i, K_i

primal-mesh cell, P_i

Reconstructing the nonlinear Coriolis force

(recall that the nonlinear Coriolis force is the the PV-flux perpendicular to the velocity)



$$\frac{\partial u_e}{\partial t} + \widehat{q}_e \left[h u \right]_e^{\perp} = \left[\nabla \left(g h_i + K_i \right) \right]_e$$
$$\frac{\partial u_e}{\partial t} + Q_e^{\perp} = \left[\nabla \left(g h_i + K_i \right) \right]_e$$

$$d_e\,Q_e^\perp = \sum_j w_e^j\,l_j\,F_j\,ar q_{ej}$$
 $F_j = \hat h_j u_j^j$ thickness flux $w_e^j = -w_j^e$ weights are equal and opposite

PV is symmetric

The nonlinear Coriolis force will be energetically neutral for any \bar{q}_{ej} . This is an extension to what Sadourny (1975) showed for regular meshes.

 $\bar{q}_{ej} = \bar{q}_{je}$

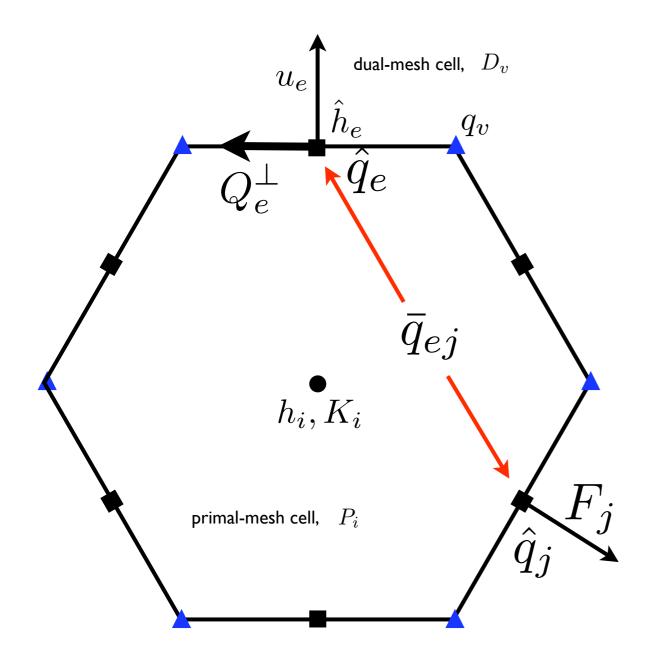
[Ref: 13, 31]





Reconstructing the nonlinear Coriolis force

(recall that the nonlinear Coriolis force is the the PV-flux perpendicular to the velocity)



$$\frac{\partial u_e}{\partial t} + \widehat{q}_e \left[h u \right]_e^{\perp} = \left[\nabla \left(g h_i + K_i \right) \right]_e$$

$$\frac{\partial u_e}{\partial t} + Q_e^{\perp} = \left[\nabla \left(gh_i + K_i\right)\right]_e$$

The curl of the above eq, leads to the below eq.

$$\frac{\partial}{\partial t} (h_v q_v) + \frac{1}{A_v} \sum_{e \in G(v)} Q_e^{\perp} dc_e = 0$$

For a uniform PV field, the above eq reduces (identically) to the vertex thickness eq.

The evolution of the discrete velocity field is compatible with the evolution of a valid, discrete PV equation. The compatibility holds to round-off.

$$\frac{\partial}{\partial t} (h_v) + \frac{1}{A_v} \sum_{e \in G(v)} F_e^{\perp} dc_e = 0$$

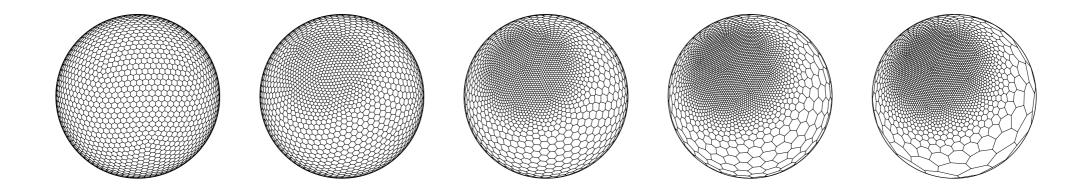


Results using the shallow-water equations.



Experimental design

- I. Every simulation is conducted with the same executable with the exact same input parameters.
- 2. There is no explicit dissipation. All numerics (with one important exception!) are ~2nd-order centered in space. The time-stepping is 4th-order Runge Kutta.
- 3. We use the anticipated potential vorticity method to dissipate potential enstrophy while conserving energy.
- 4. We will use shallow-water test cases #2 and #5 as the basis for evaluation.

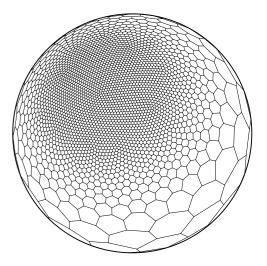


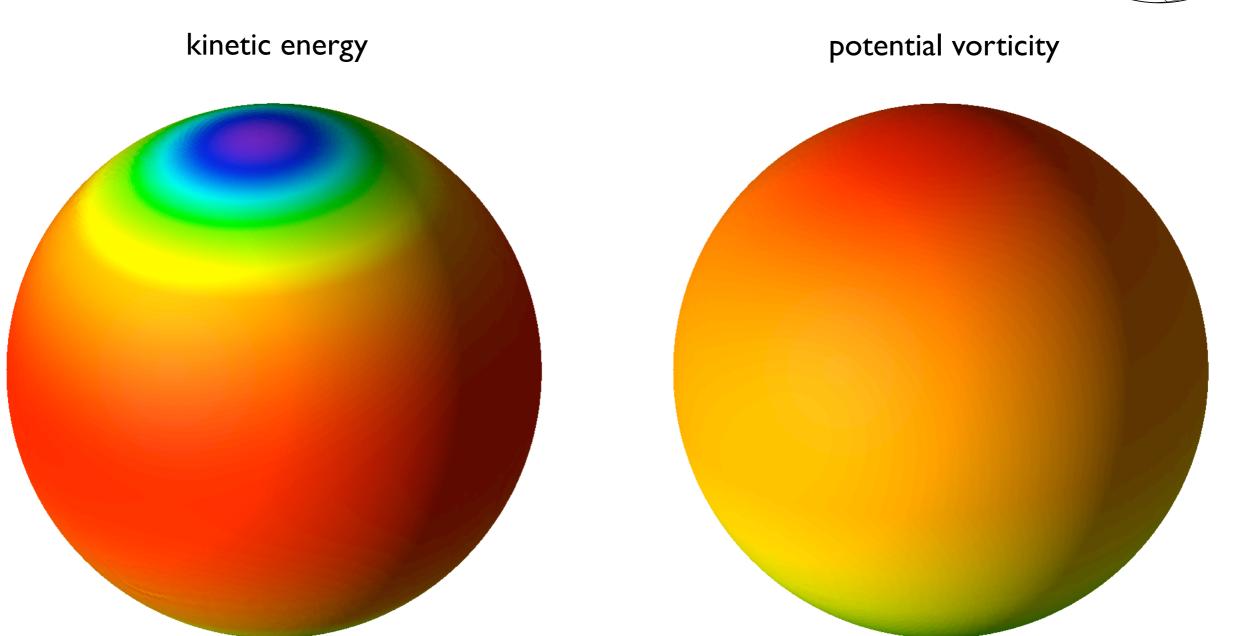




Let's start with a sanity check

x 16, 163842 nodes, shallow-water test case #2, day 50 (20 km / 320 km resolution)



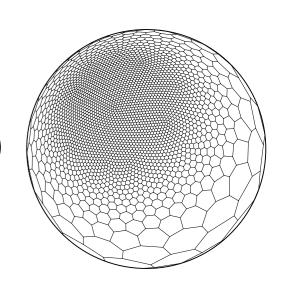


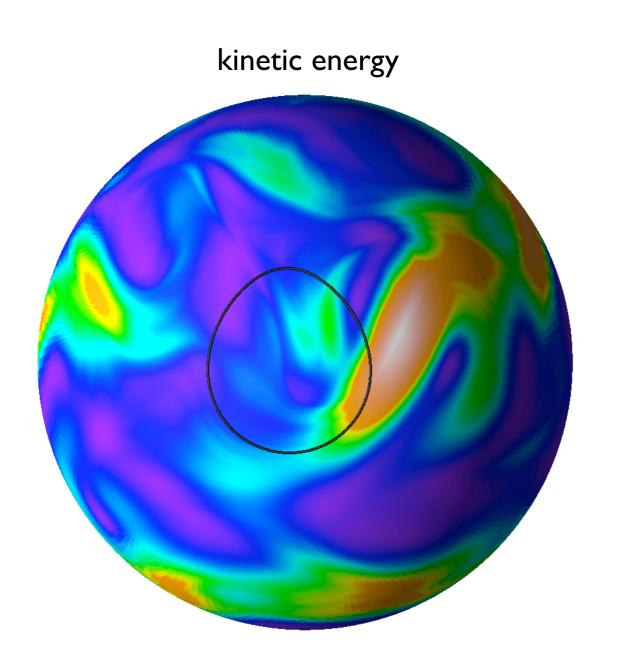


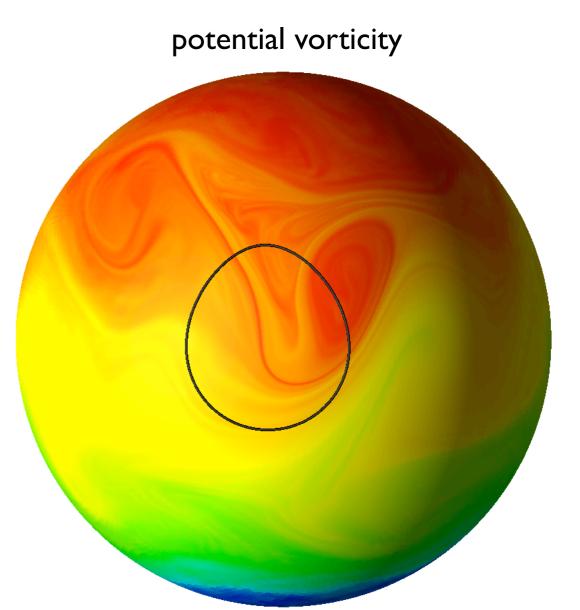


Let's start with a sanity check

x 16, 163842 nodes, shallow-water test case #5, day 50 (20 km / 320 km resolution)

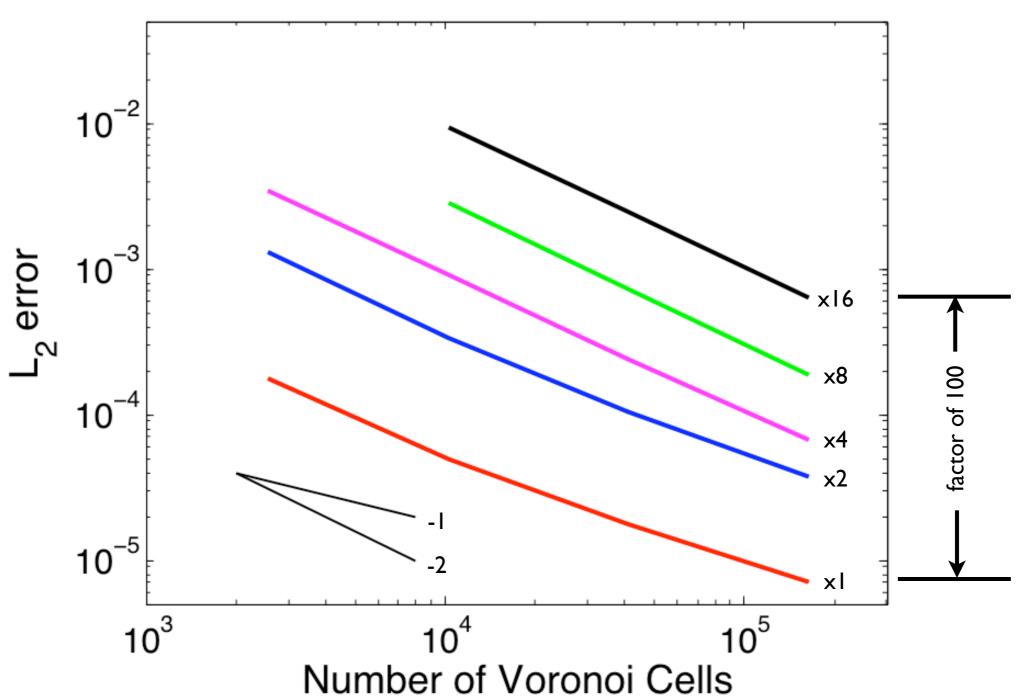






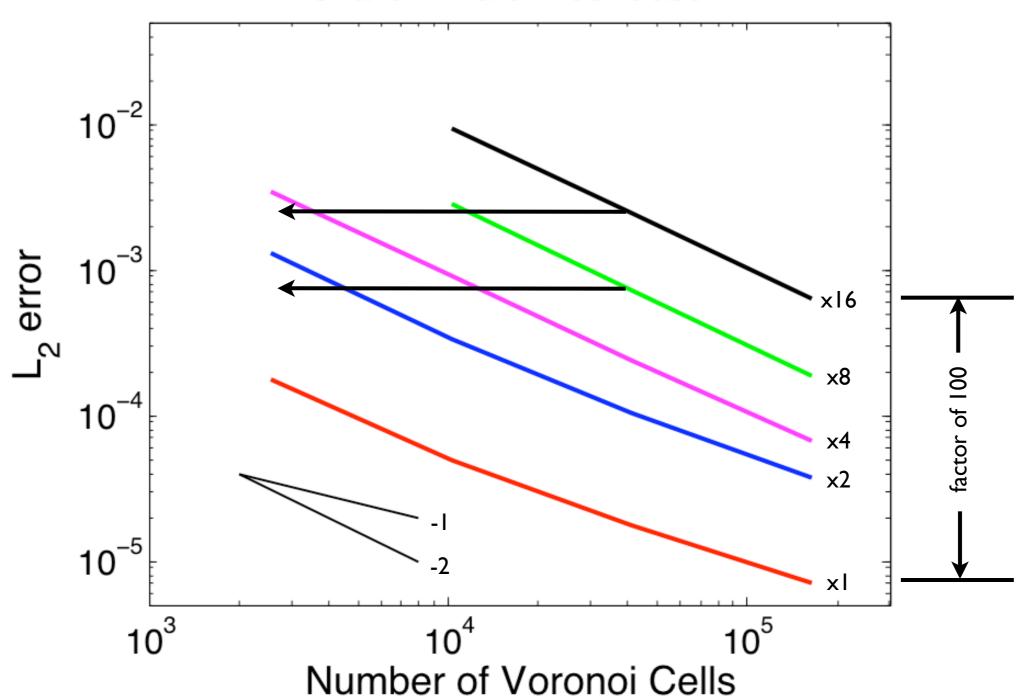
Now something a bit more quantitative ... L2 error of SWTC#2

Shallow Water Test Case 2



Now something a bit more quantitative ... L2 error of SWTC#2

Shallow Water Test Case 2



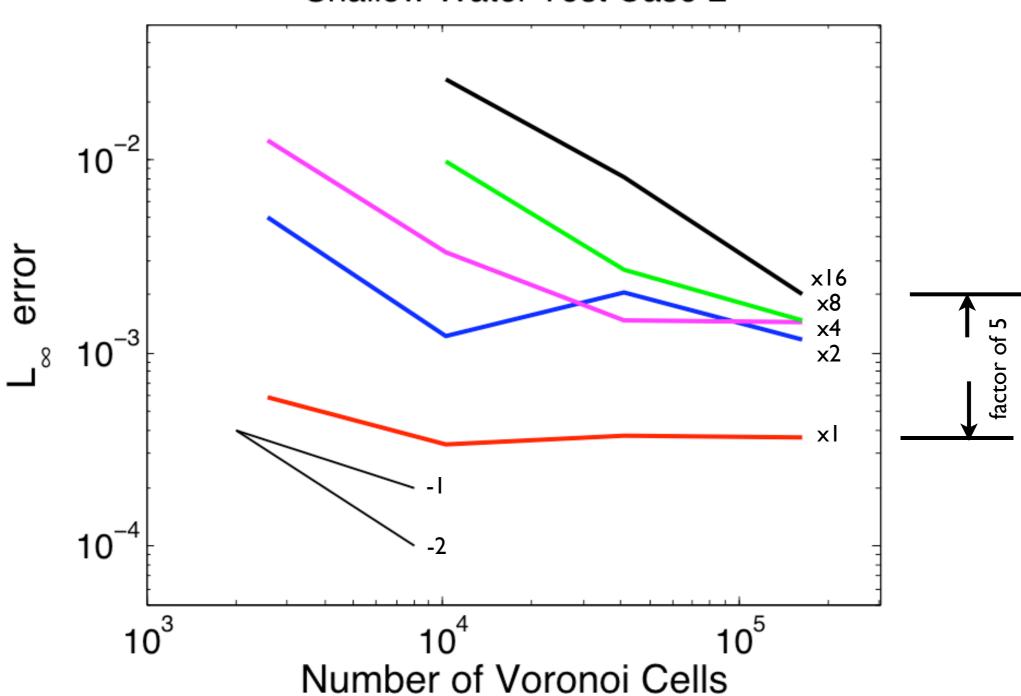
The low resolution region of the x8 40962 mesh is about the same as the resolution of the x1 2562 mesh.





Linf error of SWTC#2

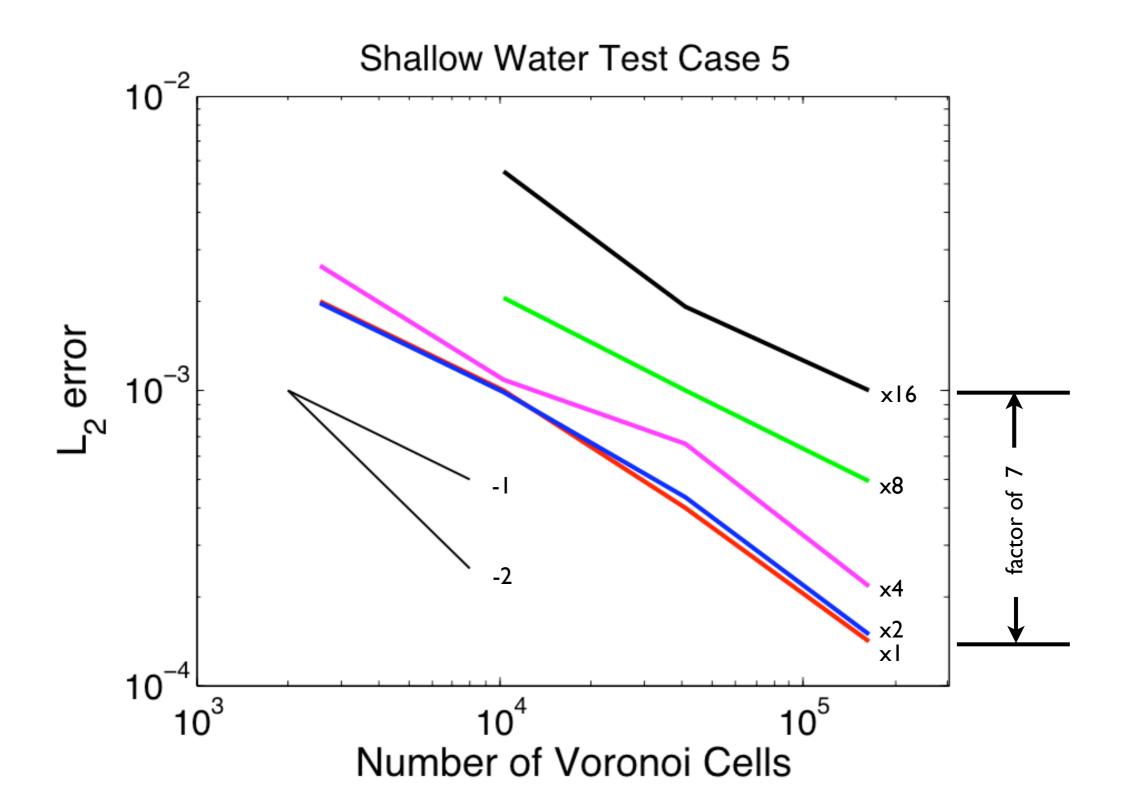








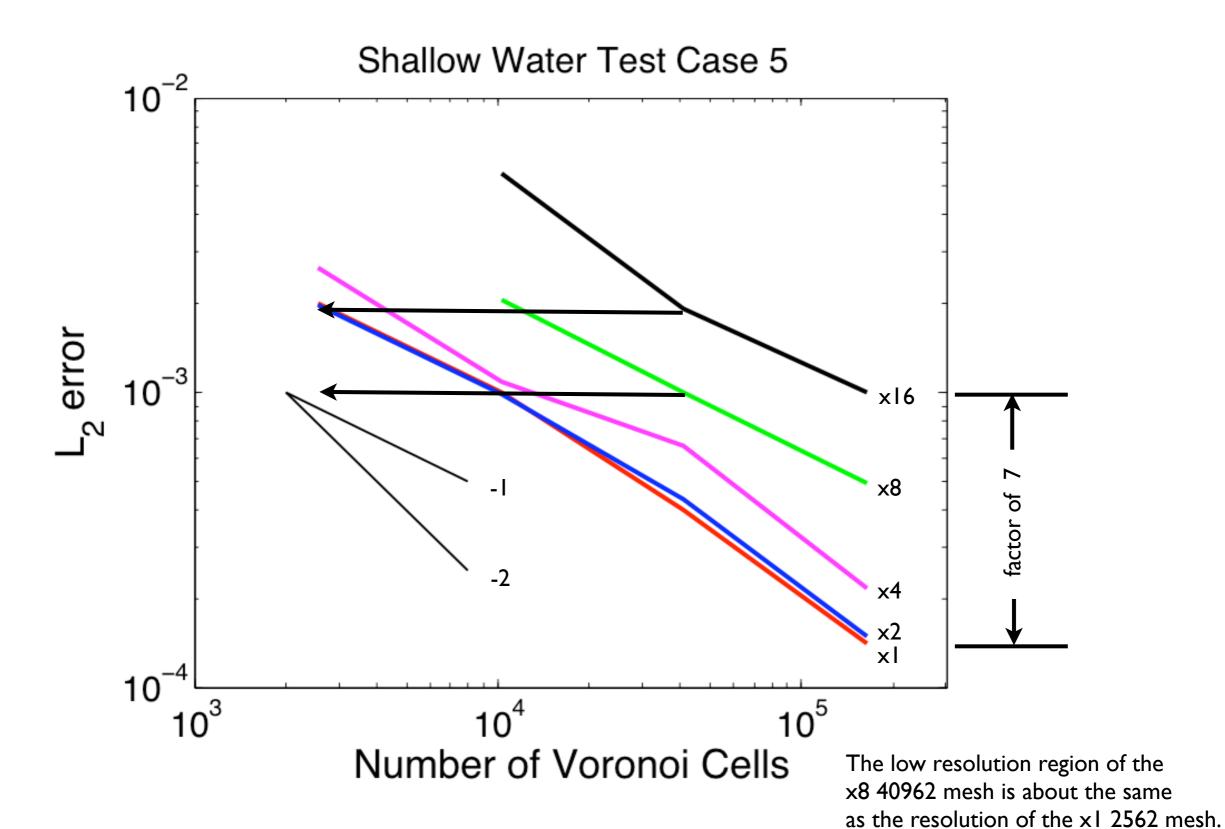
L2 error of SWTC#5







L2 error of SWTC#5

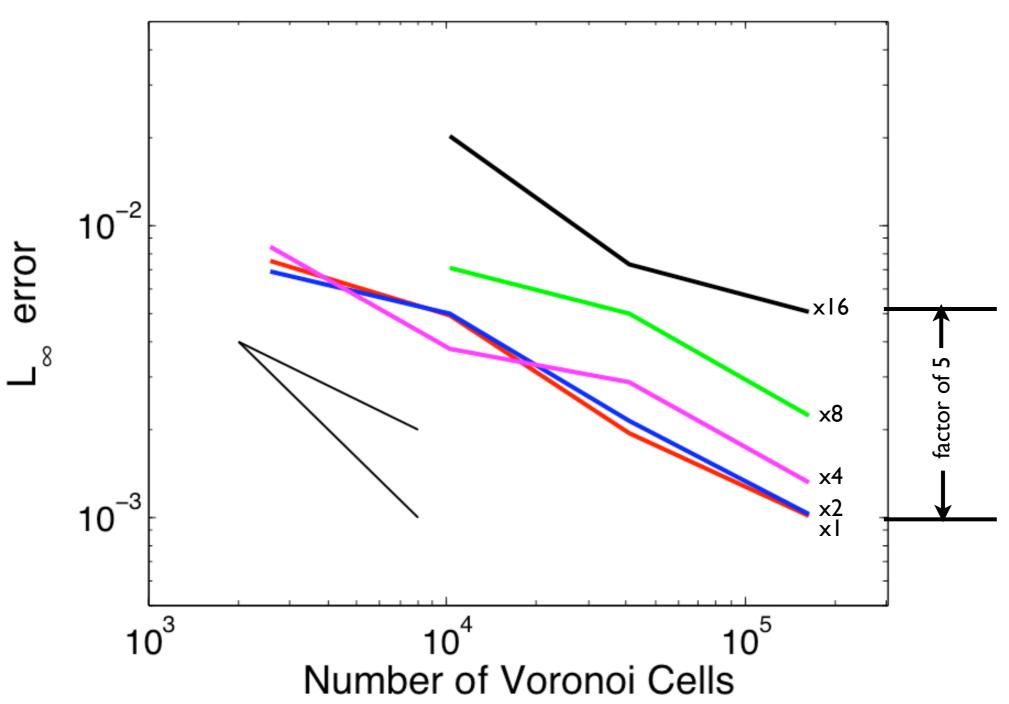






Linf error of SWTC#5

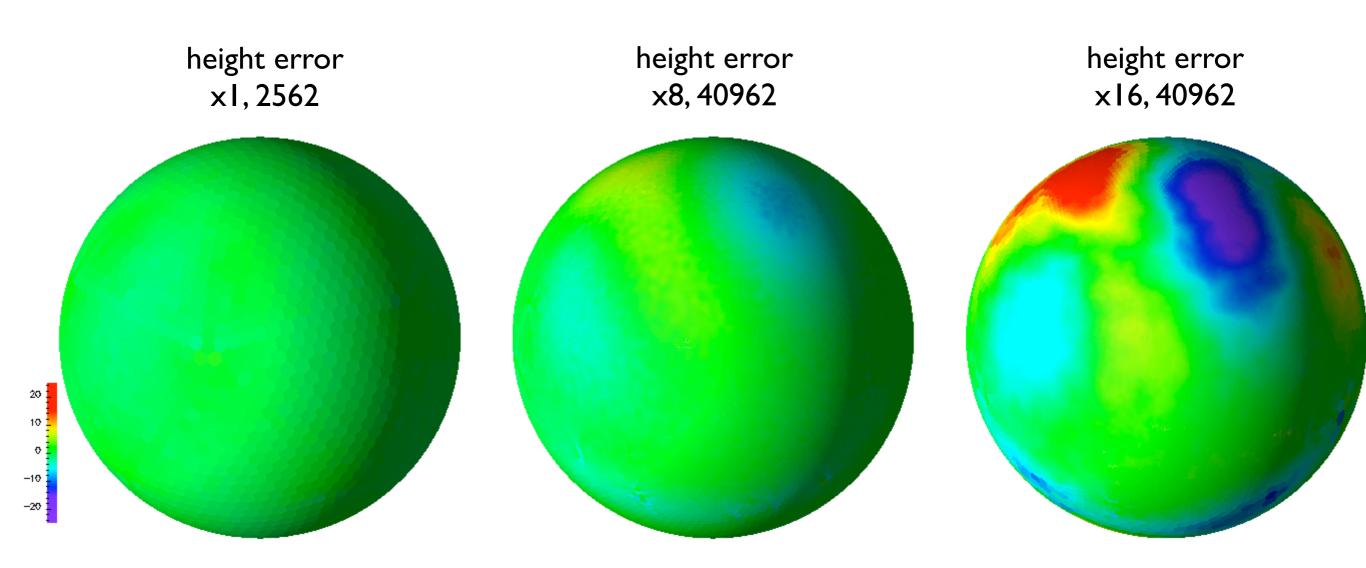








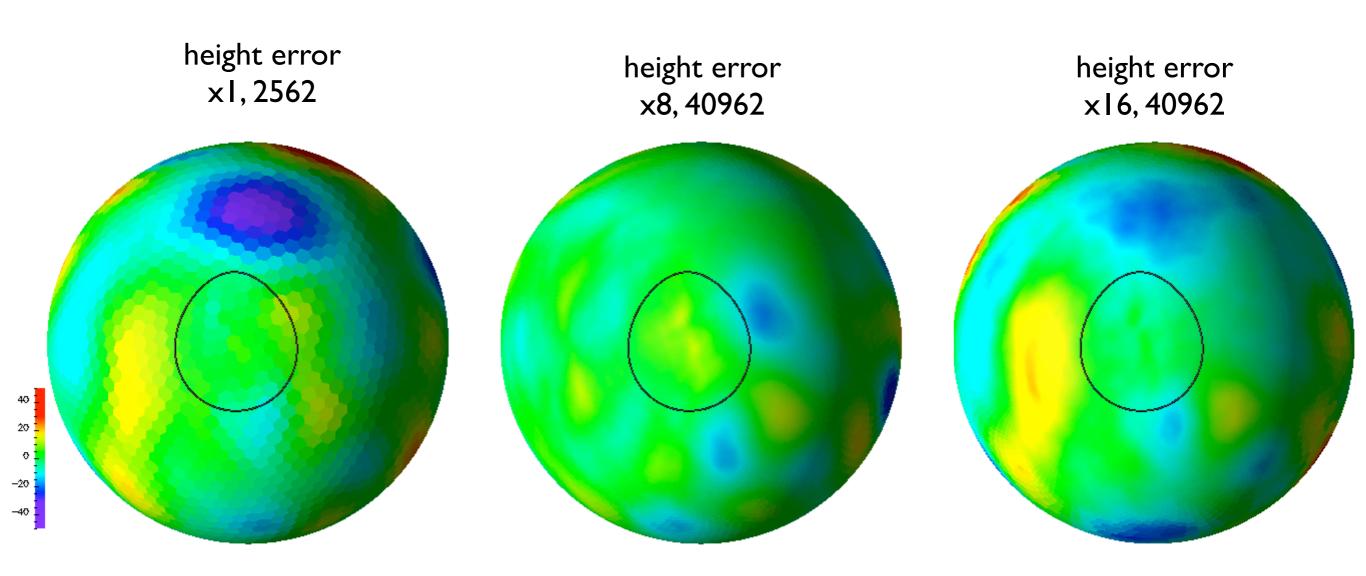
A look at the error distribution in TC2 at day 12







A look at the error distribution in TC5 at day 15







Summary of our findings to date using the variable resolution SCVTs for the simulation of the shallow-water equations.

- 1. The numerical algorithm is robust for very large ratios in grid resolution without the need for ad hoc stabilization methods.
- 2. The relevant global invariants (mass, potential vorticity, energy) are conserved regardless of the mesh is of variable resolution or uniform.
- 3. As the mesh variation increases, the operator truncation error and the PDE solution error raises.



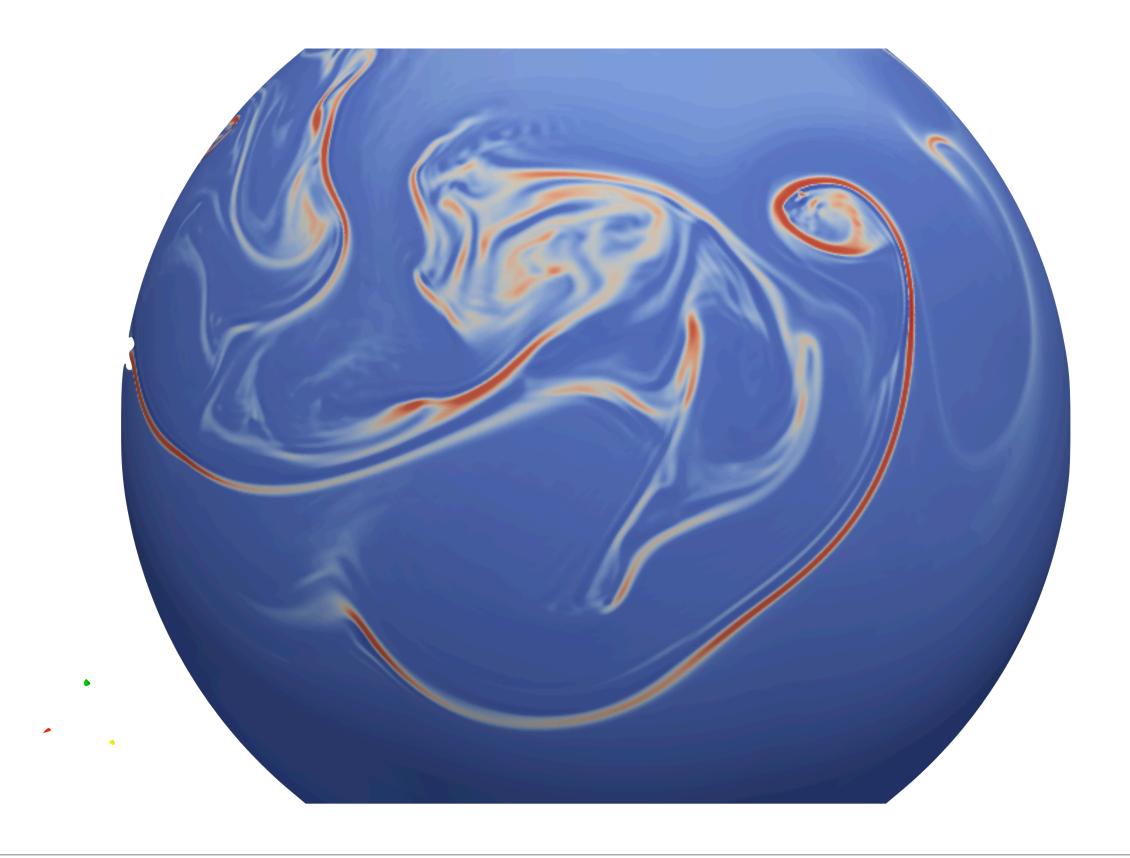
Results using the global atmosphere model.

The global, hydrostatic atmosphere model uses a hybrid sigma coordinate in the vertical direction with the vertical discretization taken largely from the NCAR Weather and Forecast Research (WRF) model.





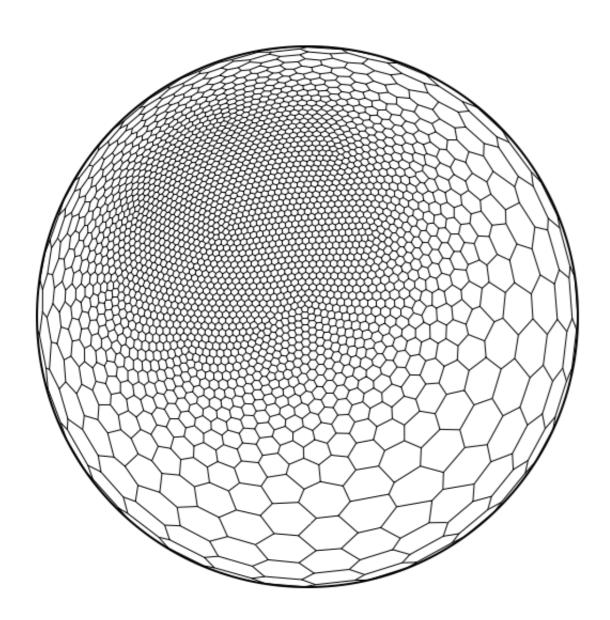
Baroclinic Eddy Test Case Day 15, Relative Vorticity, Global 15 km mesh



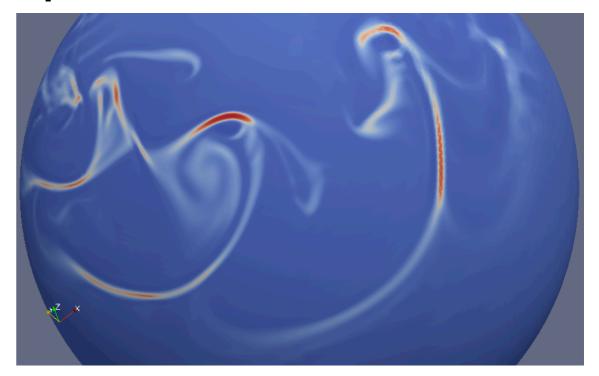




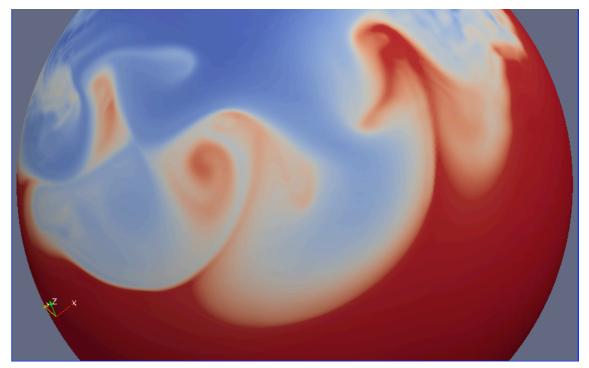
Solving the primitive equations: a global hydrostatic atmosphere simulation.



This is the Jablonowski and Williams baroclinic eddy test case on a mesh with 40 km grid spacing in high resolution zone and 320 km grid spacing in low resolution zone.



relative vorticity



potential temperature





Results using the global ocean model.

The global, hydrostatic ocean model can be run as a z-level model or as a layered isopycnal model.



Ocean Double-Gyre Problem

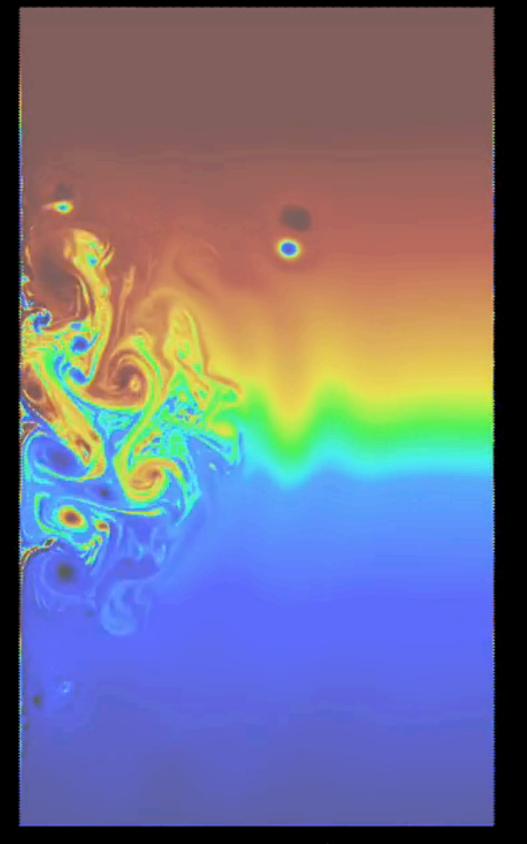
5 km mesh
5000 km x 2500 km domain
Forcing consistent with Greatbatch and Nadiga (2000).
No bottom drag
Laplacian mixing on velocity at 1.0 m²/s

NOTE: The purpose here is to confirm that the numerical scheme can handle strong eddy activity without the need for stabilization through ad hoc dissipation. The scheme is stable for a very wide range of dissipations, even those that clearly leace to an underdamped system.

Potential Vorticity: one frame per day







Potential Vorticity: one frame per day

http://public.lanl.gov/ringler/movies/2009/eddies_18.mov

Los Alamos NATIONAL LABORATORY

Ocean Double-Gyre Problem

5 km mesh
5000 km x 2500 km domain
Forcing consistent with Greatbatch and Nadiga (2000).
No bottom drag
Laplacian mixing on velocity at 1.0 m²/s no-slip boundary conditions

NOTE: The purpose here is to confirm that the numerical scheme can handle strong eddy activity without the need for stabilization through ad hoc dissipation. The scheme is stable for a very wide range of dissipations, even those that clearly lead to an underdamped system.

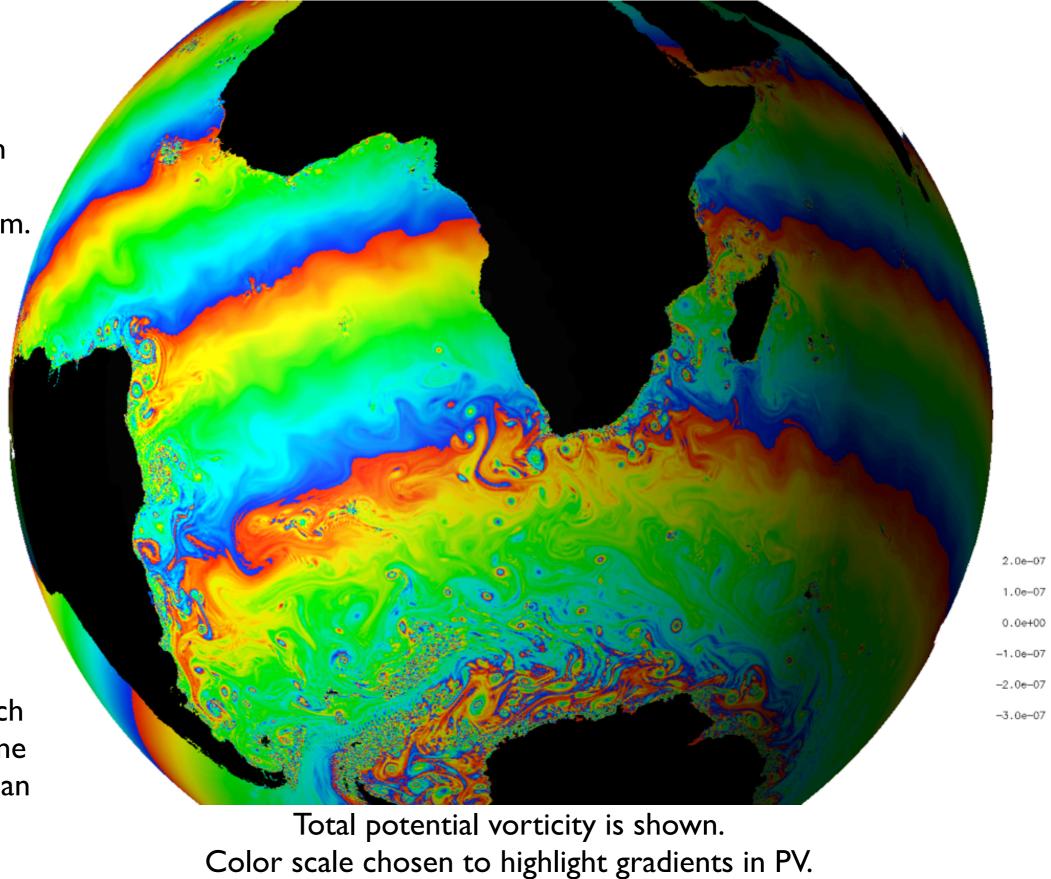


An application to global ocean modeling

Uniform and variable resolution Voronoi tessellations have been created at resolutions between 120 and 15 km.

Right: Global 15km mesh, PV shown, single-layer, flat bottom.

The meshes use nearest-neighbor search from ETOPO2 to define land/sea mask and ocean depth.



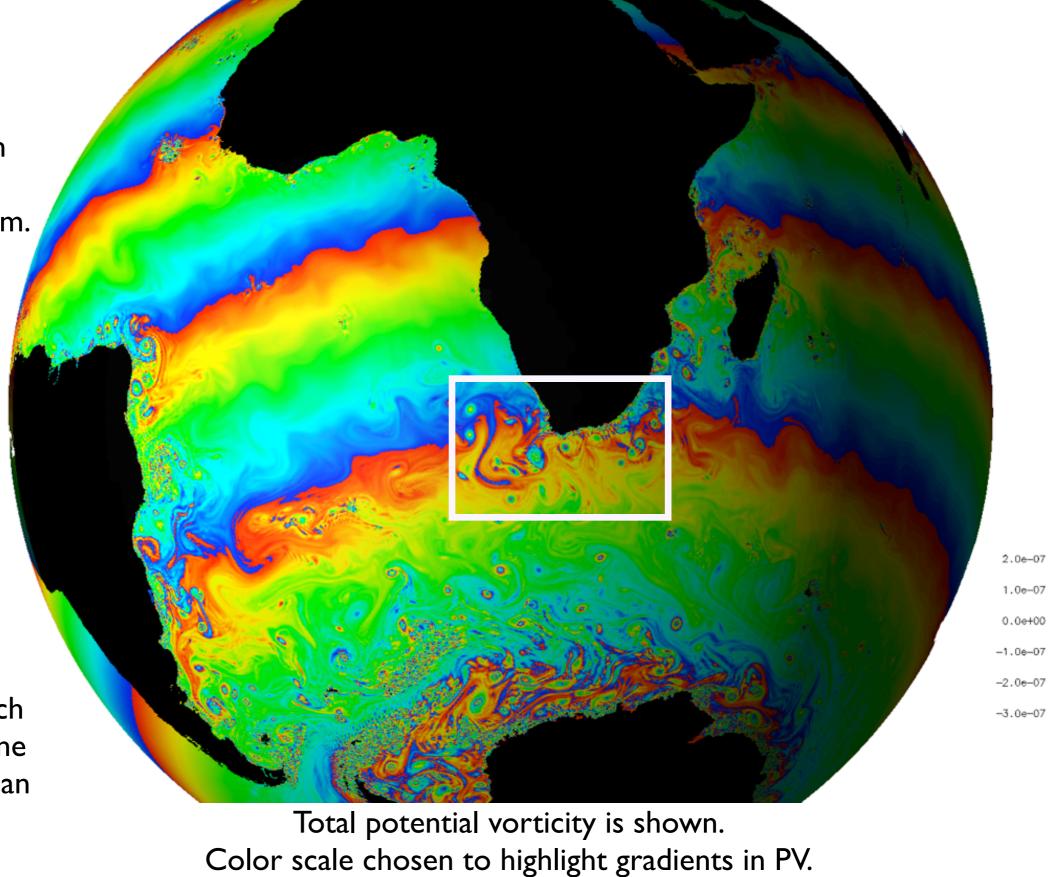


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MPAS Ocean Model:

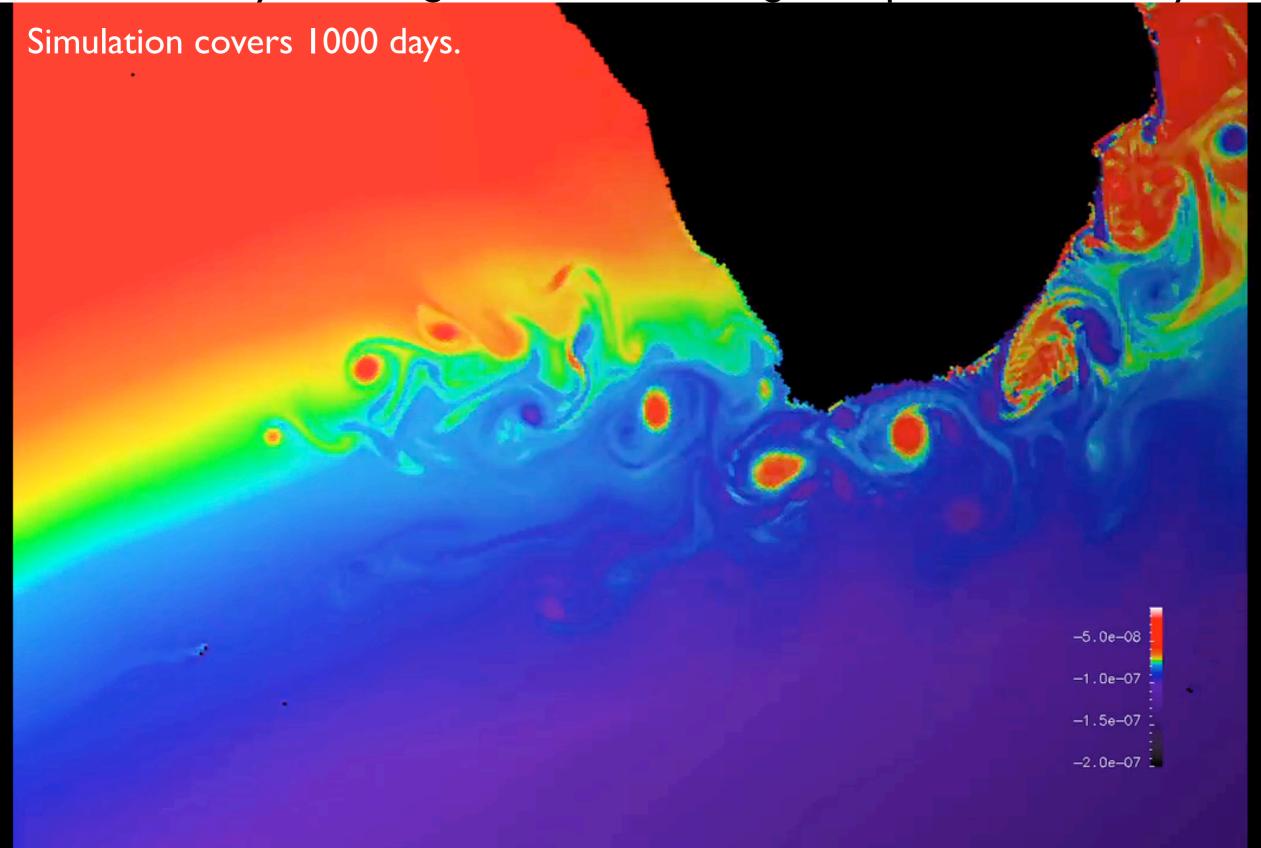
Global eddy-resolving simulation showing total potential vorticity.





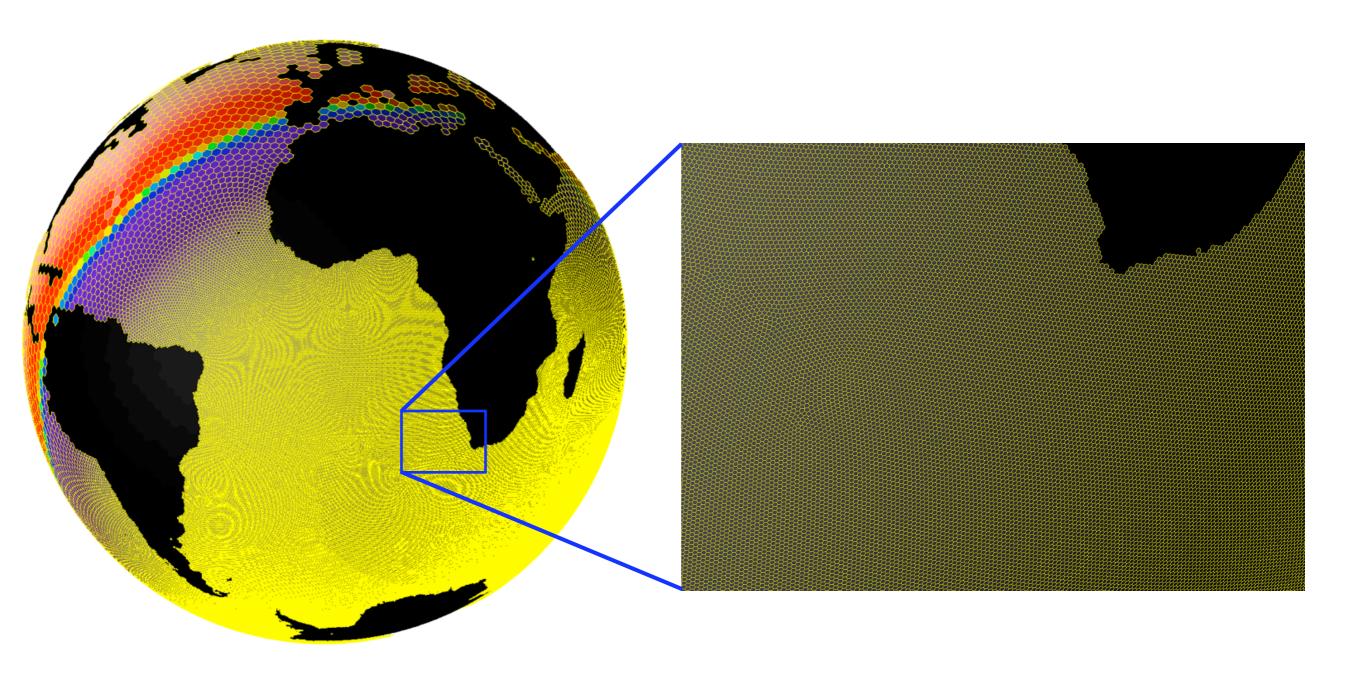
MPAS Ocean Model:

Global eddy-resolving simulation showing total potential vorticity.





Ocean modeling with variable resolution meshes

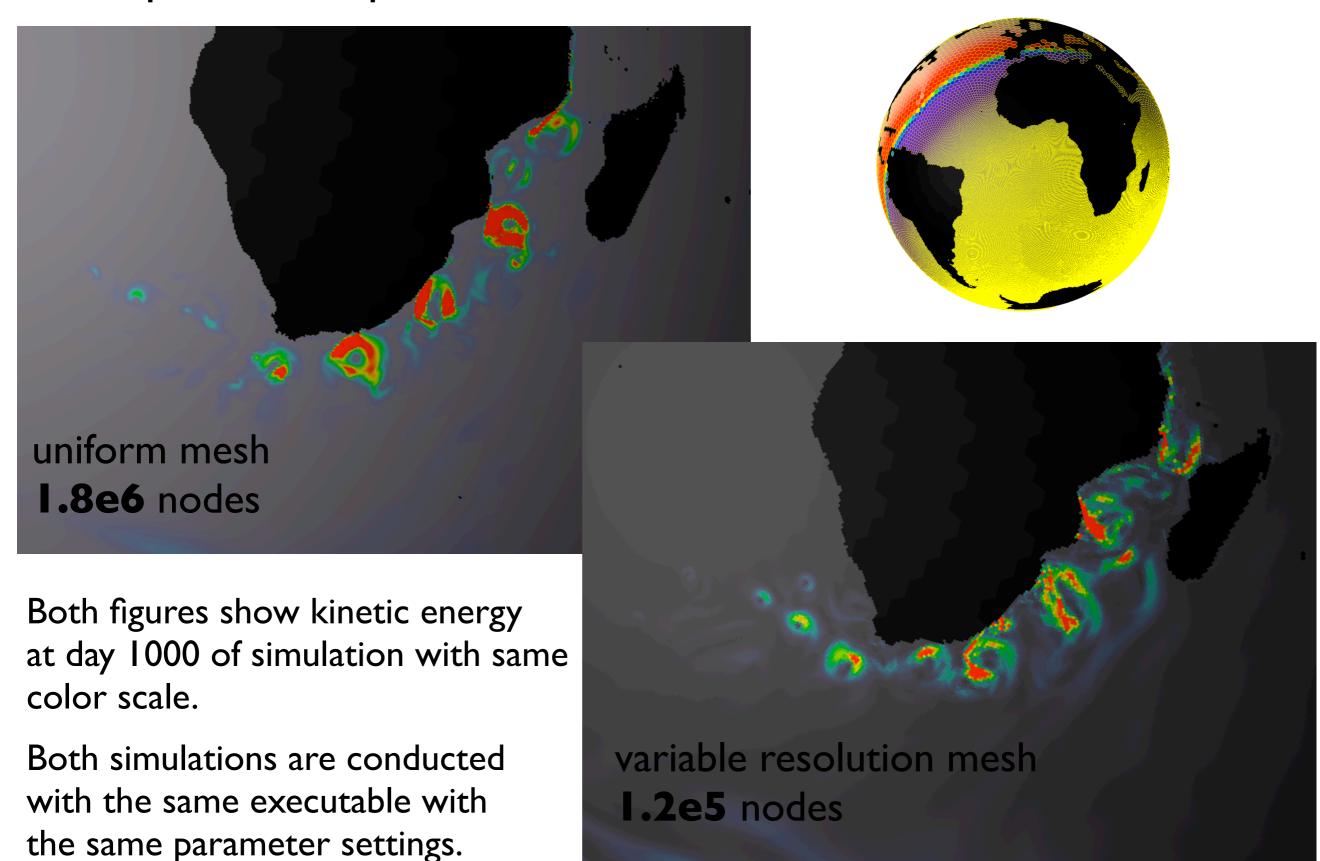


Mesh grid spacing varies by a factor of 8, ranging from 25 km to 200 km. This mesh has about 1/10 the number of nodes as the globally uniform 15 km mesh.





Comparison of quasi-uniform to variable-resolution simulation.



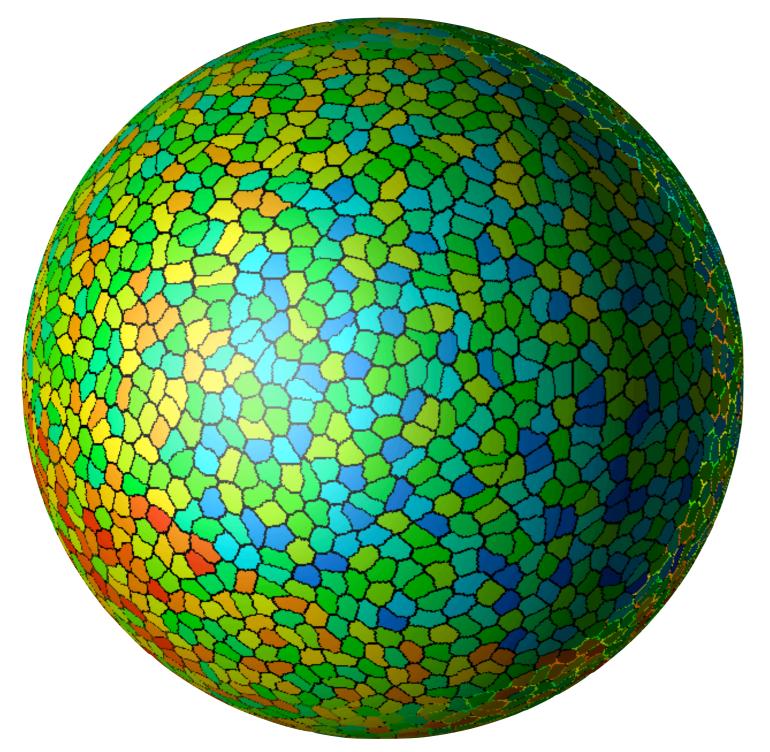




Computational considerations.

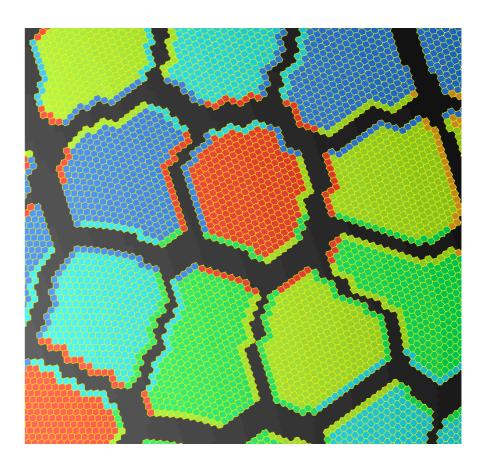


Domain decomposition is done using ParMetis.



Each block is assigned to a node.

In order to support variableresolution meshes, we are required to use an unstructured grid in the horizontal directions.



Each block includes a halo region of arbitrary width.





An early look at the strong scaling of the prototype system.

An analysis of the domain decomposition strategy in comparison to its peers (LANL POP and SNL HOMME) indicates that we should expect strong scaling out to O(100K) nodes with a global mesh resolution of approximately 10 km.



Lobo: AMD Opteron cores for computation, using an Infiniband interconnect. 272x16-core nodes for production capacity computing. Each node has 32 GB of non-uniform-access memory. CPUs run at 2.2 Ghz and have 0.5-MB L2.





An idea codified

Model for Prediction Across Scales: A comprehensive multi-scale approach to climate modeling

MPAS is a joint NCAR/LANL model development effort.

NCAR is developing a global non-hydrostatic (regionally cloud resolving) atmosphere model.

LANL is developing a global hydrostatic (regionally eddy resolving) ocean model

LANL is developing a (regionally ice-stream resolving) ice sheet model.

All MPAS models are build for a common software framework that handles memory allocation, domain decomposition, I/O.





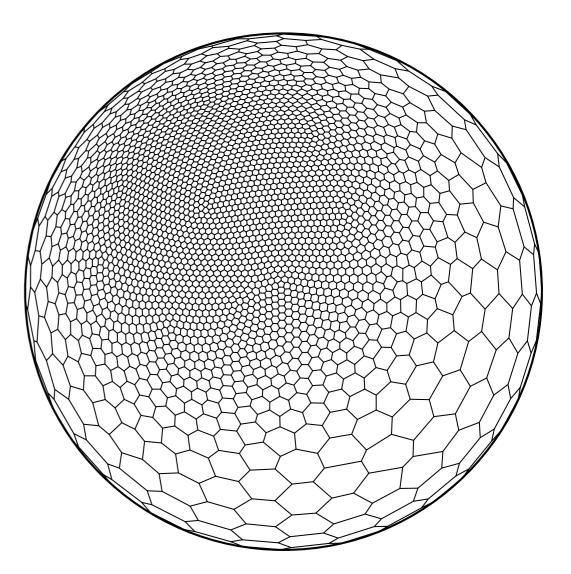
MPAS is a new approach to climate system modeling.

At a minimum it will be disruptive. At a maximum it will be transformative.

Looking forward:

A comprehensive multi-scale capability will be characterized by three attributes:

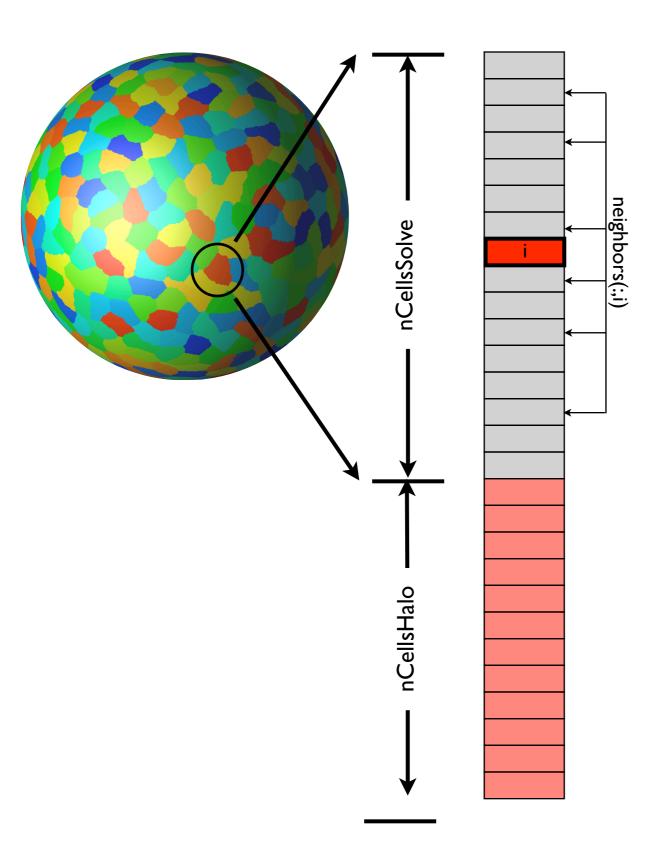
- 1. **Multi-resolution**: The ability to resolve different spatial and temporal scales in different parts of the global domain.
- 2. **Scale-aware closures**: The ability to transition seamlessly from modeling a process directly through simulation to modeling a process indirectly through parameterization.
- 3. **Multiple governing equation sets**: The ability to use the appropriate (and different) equation sets in different parts of the global domain.



Challenges in constructing a true multi-scale climate system model.



Challenge #1: Computational Efficiency



All neighbor data required to compute spatial derivatives and averages is done via indirect addressing using, for example, neighbors(:,i).

We attempt to mitigate these non-uniform data access patterns in two ways:

- I. The ordering of the cells is determined, for example, by Reverse Cuthill-McKee (RCM) to maximize cache reuse.
- 2. The data access pattern is repeated in the vertical, leading to arrays dimensioned as mass(nVertLevels, nCellsTotal, nBlocks).

Still, we have a significant challenges in competing with models based on uniform data access patterns and will have to reconsider all aspects of our algorithms.



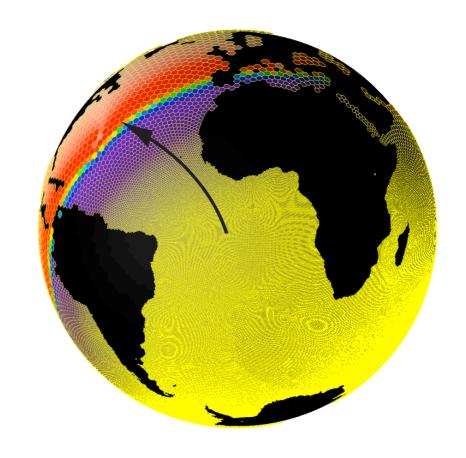


Challenge #2: Scale-Aware Parameterizations

We have generated a modeling approach that allows eddies to be resolved regionally.

By construction we have designed a system where important phenomena are resolved in some locations but not in others.

As a result, we obligated to develop scale-aware parameterizations that can seamlessly "turn-off" as the underlying phenomenon become resolved.



ocean sub-mesoscale eddies **resolved**

ocean sub-mesoscale eddies marginally resolved

ocean sub-mesoscale eddies **parameterized**

ocean mesoscale eddies fully **resolved**

ocean mesoscale eddies resolved

ocean mesoscale eddies **parameterized**

0.1 km

I km

10 km

100 km

mesh resolution



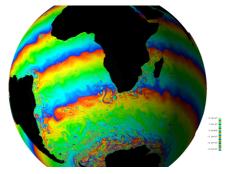


Postdoctoral Research Positions Related to MPAS



Postdoctoral Research Positions

Multi-scale Climate System Model Development Los Alamos National Laboratory



The Climate, Ocean and Sea-Ice Modeling (COSIM) group at LANL is expecting to open up to four postdoc positions in August 2010 for the development of atmosphere, ocean and land-ice components within our Model for Prediction Across Scales (MPAS) project.

MPAS is a joint project between LANL and the National Center for Atmospheric Research (NCAR) to prototype, evaluate and make scientific use of multi-scale components used in climate system models, including atmosphere, ocean and land-ice components.

Position #1: Evaluation of Regional Climate Modeling Frameworks: Atmosphere Focus

This postdoc position will focus on the evaluation of MPAS for the regional modeling of atmosphere climate dynamics. Work will include the evaluation of the MPAS variable-resolution approach as compared to alternative approaches based on global high-resolution and limited-area modeling. In addition, an evaluation of simulations with full physics will be conducted. This postdoc will work closely with the mesoscale modeling group at NCAR and will be strongly encouraged to visit NCAR frequently.

Position #2: Evaluation of Regional Climate Modeling Frameworks: Ocean Focus

This postdoc position will focus on the evaluation of MPAS for the regional modeling of ocean climate dynamics. Work will include an evaluation of MPAS to regionally simulate ocean eddy processes. Special attention will be given to developing scale-aware closures for parameterizing ocean eddy activity.

Position #3: Regional Climate Modeling to Assess Impact of Vegetation Change

This postdoc position will focus on the coupling of the MPAS global hydrostatic atmosphere model to the NCAR Community Earth System Model in order to study the regional impact of vegetation change due to anthropogenic greenhouse gases. The postdoc researcher will work within a strong interdisciplinary team that include expertise in dynamic vegetation modeling, plant physiology, data assimilation and remote sensing.

Position #4: Development of a Variable-Resolution Land Ice Model

The postdoc position will focus on the develop of a variable resolution land ice model using the standard highorder approximation approach. This postdoc position will likely be located at Florida State University under the guidance of Max Gunzburger. The postdoc researcher will be given a "visiting scientist" position at LANL to facilitate frequent and extended stays with the COSIM group.

Qualifications: All positions require strong training in applied mathematics and/or computational physics. Requirements also include a demonstrated ability to publish peer-reviewed papers, effective written and oral communication skills, willingness to work in a team environment, and a Ph.D. pending or received within the last five years. A demonstrated ability to develop, analyze and implement numerical methods for geophysical applications is also highly desired. In addition, desired skills include a background in atmospheric or ocean dynamics and working knowledge of FORTRAN, C or C++.

This is an informal advertisement. All postdoc positions may not be filled. Interested students should send CV and one page research statement to ringler@lanl.gov with subject title "MPAS Postdoc". Once the positions are officially opened, qualified applicants will be asked to submit a formal application to the LANL. Exceptional candidates will be consider for the prestigious LANL Director's Postdoc Program.

We have a variety of opportunities to design, develop and interrogate this multi-scale approach to climate system modeling.

The first two positions are primarily model development and numerical analysis positions to study to utility of this multi-scale approach relative to global high-resolution and limited-area modeling.

The third position is focused on the use of the MPAS approach to study regional climate change and vegetation dynamics.

The final position is for the design, development and testing of a multi-scale ice-sheet model.

All work is expected to support the NCAR Community Earth System Model effort.

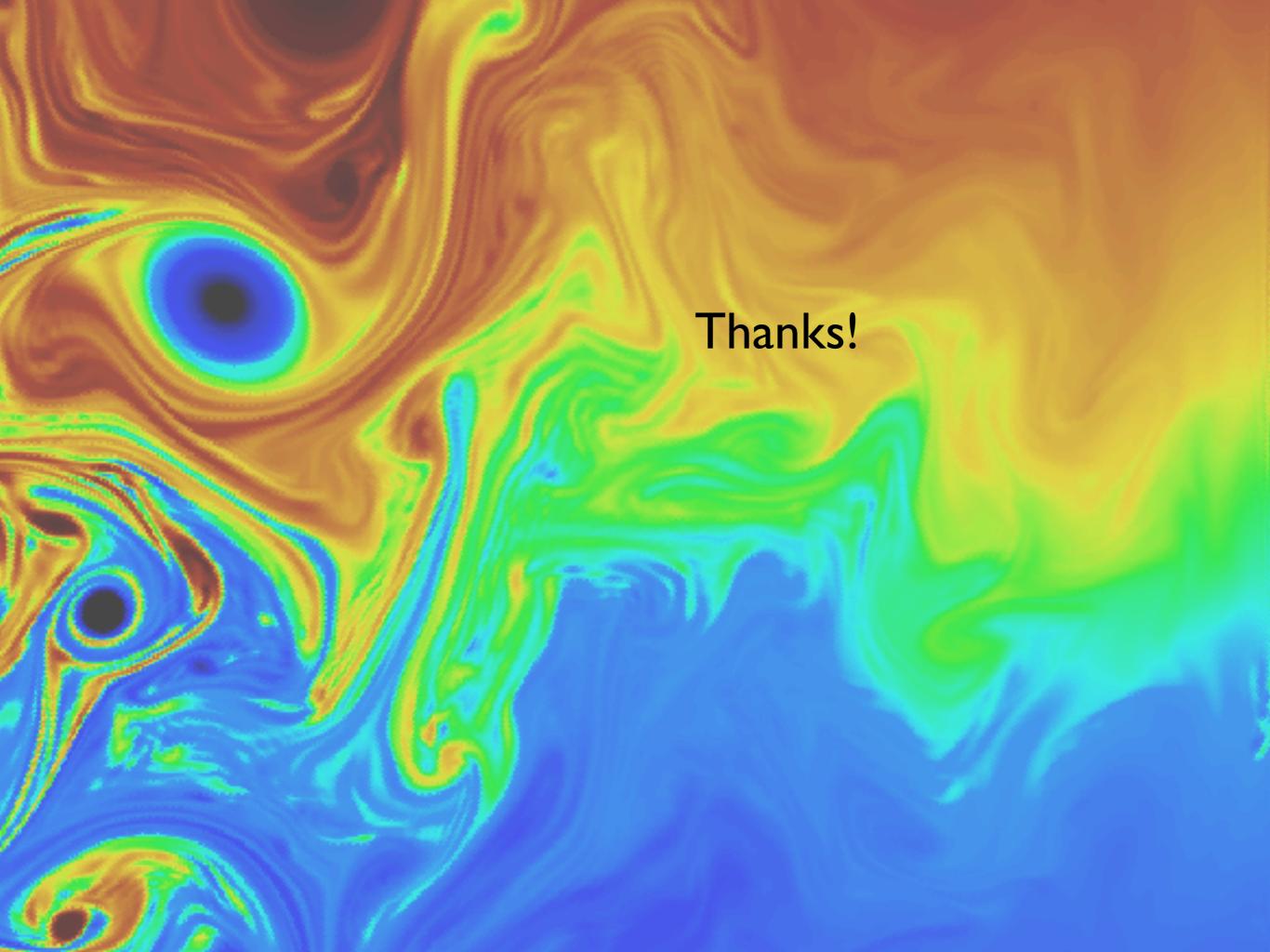




Overall Summary

- I. Global climate modeling is (and will likely always be) an under-resolved endeavor. While this is great job security for climate modelers, it means that the community will be continually challenged to develop new ways to model this system.
- 2. The community would greatly benefit from the creation of a comprehensive, multi-scale approach to global climate modeling.
- 3. The MPAS project is exploring one approach to multi-scale modeling based on variable resolution meshes combined with scale-aware parameterizations.





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